Cumulative Review from Gravitation to Faraday's Law

Gravitation

Gravitational force:

$$\vec{F}_g = G\frac{mM}{r^2}(-\hat{r})$$

Gravitational potential energy:

$$U(r) = -G\frac{Mm}{r}$$

Angular momentum (circular motion):

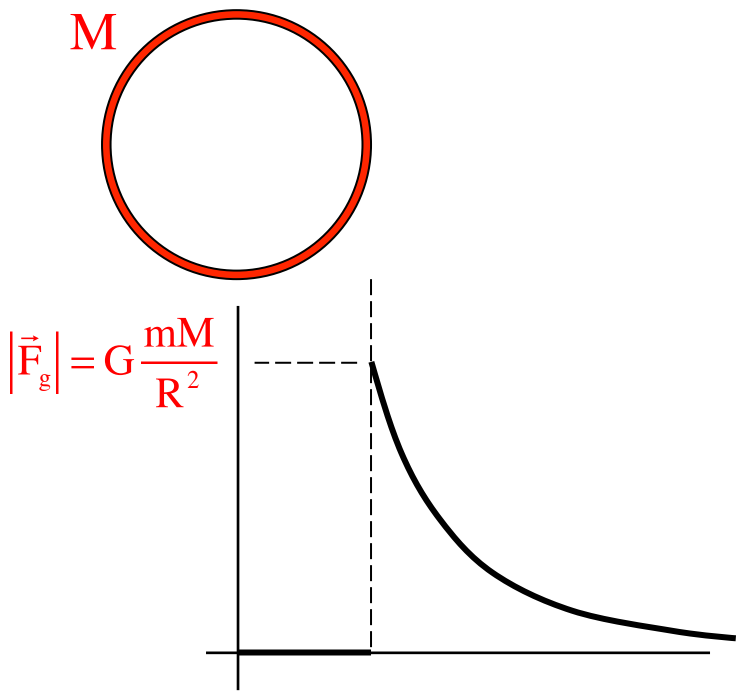
$$L = I\omega$$
$$= \left(mR^2\right)\left(\frac{v}{R}\right) = mvR$$

Total mechanical energy (circular motion):

$$E_{total} = \frac{1}{2}mv^2 + \left(-G\frac{mM}{r}\right) \qquad \left(\text{but } G\frac{mM}{r^2} = m\frac{v^2}{r} \quad \Rightarrow \quad v^2 = G\frac{M}{r}\right)$$
$$= \frac{1}{2}m\left(G\frac{M}{r}\right) + \left(-G\frac{mM}{r}\right) = -\frac{1}{2}G\frac{mM}{r}w$$

Graphs

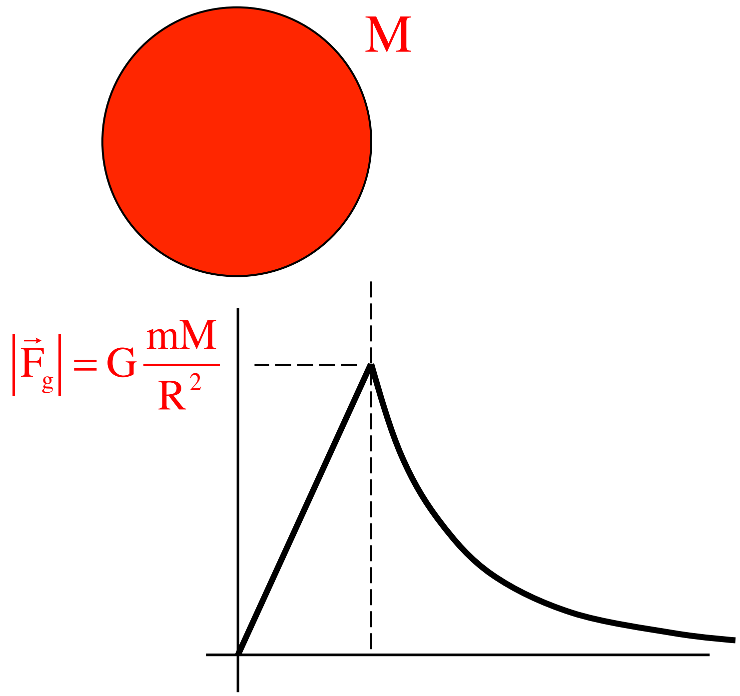
Magnitude of force on particle due to a spherical shell:



$$\left|\vec{F}_g\right| = 0 \qquad \text{for } r < R$$

$$\left|\vec{F}_g\right| = G\frac{mM}{r^2} \qquad \text{for } r \geq R$$

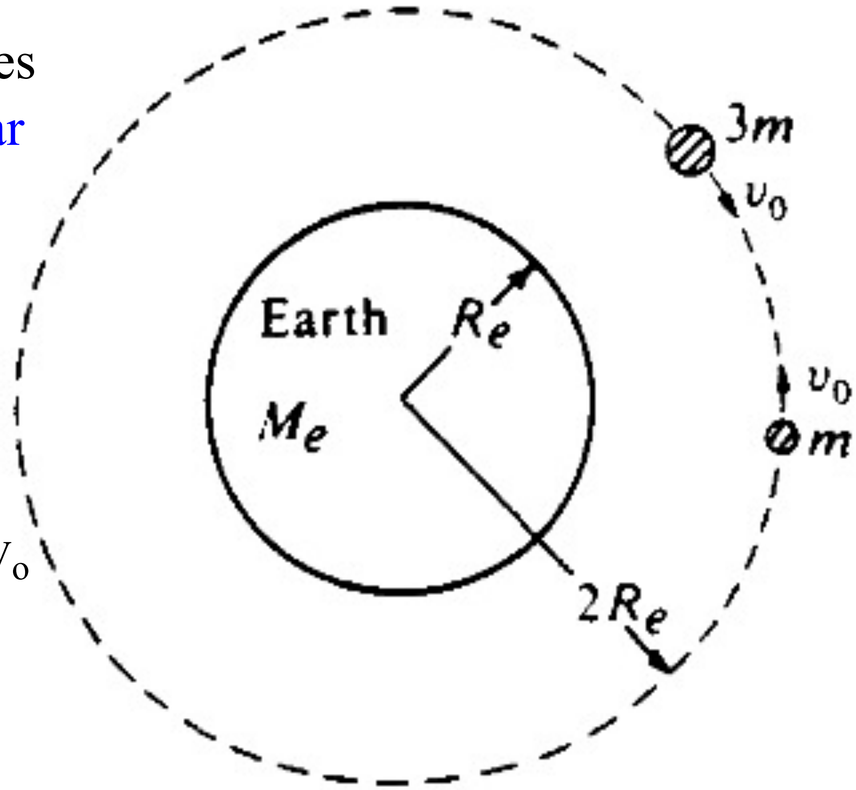
Magnitude of force on particle due to a solid shell:



$$\left|\vec{F}_g\right| = Gm\left(\frac{M}{R^3}\right)r \qquad \text{for } r < R$$

$$\left|\vec{F}_g\right| = G\frac{mM}{r^2} \qquad \text{for } r \geq R$$

AP Example 7: Two satellites of masses m and 3m, respectively, are in the same circular orbit about the Earth's center, as shown in the diagram above. The Earth has mass M_e and radius R_e. In this orbit, which has a radius of $2R_e$, the satellites initially move with the same orbital speed v_o but in opposite directions.



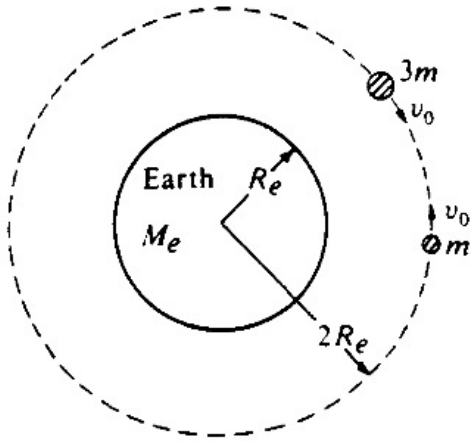
a.) Derive an expression for the orbital speed v_o of the satellites in terms of G, M_e, and R_e.

$$\sum F_{radial}:$$

$$G\frac{M_e \cancel{m_1}}{(2R_e)^2} = \cancel{m_1}\left(\frac{{v_o}^2}{2R_e}\right)$$

$$\Rightarrow \quad v_o = \left(G\frac{M_e}{2R_e}\right)^{1/2}$$

Using conservation of momentum:



$$\sum p_1 + \sum F_{ext}\Delta t = \sum p_2$$

$$\left[-(3m)v_o + mv_0\right] + 0 = -(4m)v$$

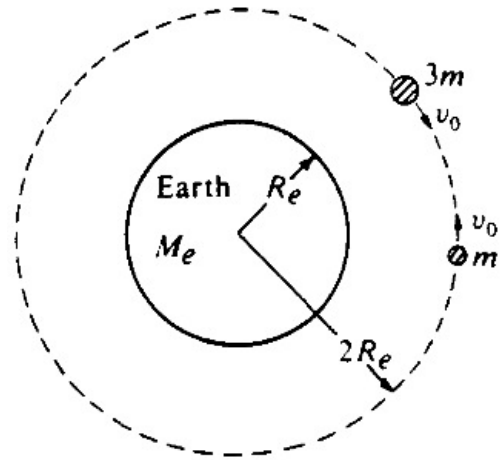
$$\Rightarrow \quad -2\cancel{m}v_o = -4\cancel{m}v$$

$$\Rightarrow \quad v = \frac{1}{2}\left(G\frac{M_e}{2R_e}\right)^{1/2}$$

where again, the negative sign suggests the final velocity is in the same direction as the original direction-of-motion of the *3m* mass.

Clearly, the *conservation of momentum* is the easier way to go here, but both are educational.

c.) Calculate the *total mechanical energy* of the system immediately after the collision in terms of G, m, M_e, and R_e.



$$E = \frac{1}{2}(4m)v^2 + \left(-G\frac{M_e(4m)}{(2R_e)}\right)$$

$$= \frac{1}{2}(4m)\left(\frac{1}{2}\left(G\frac{M_e}{2R_e}\right)^{1/2}\right)^2 + \left(-G\frac{M_e(4m)}{(2R_e)}\right)$$

$$= \left(\frac{1}{8}\left(G\frac{M_e(4m)}{2R_e}\right)\right) + \left(-G\frac{M_e(4m)}{(2R_e)}\right)$$

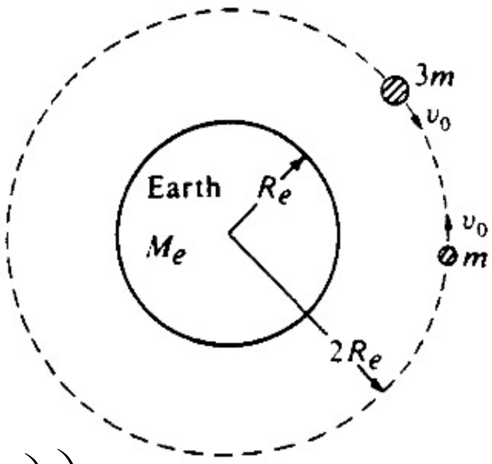
$$= -\frac{7}{8}\left(G\frac{M_e(4m)}{2R_e}\right)$$

As the gravitational potential energy is $U = \left(-G\frac{M_e(4m)}{(2R_e)}\right)$

apparently:

$$E = \frac{7}{8}U$$

Note: For this new combo-satellite to carry the new velocity in a circular orbit, its new radius would have to be:

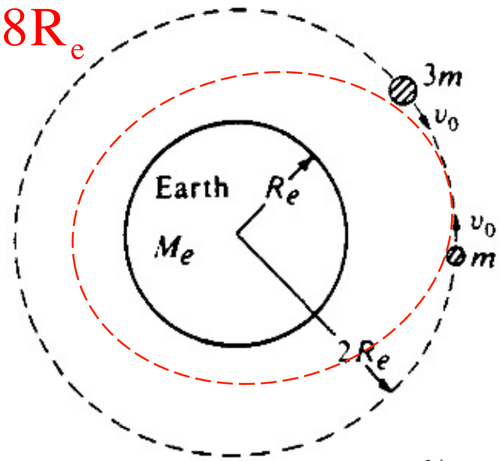


$$G\frac{M_e(4m)}{(r_{new})^2} = (4m)\left(\frac{v_{new}^{\ 2}}{r_{new}}\right)$$

$$G\frac{M_e(4m)}{(r_{new})^2} = (4m)\left(\frac{\left(\frac{1}{2}\left(G\frac{M_e}{2R_e}\right)^{1/2}\right)^2}{r_{new}}\right) = (4m)\left(\frac{\left(\frac{1}{4}G\frac{M_e}{2R_e}\right)}{r_{new}}\right) = G\frac{mM_e}{2R_e r_{new}}$$

$$G\frac{M_e(4m)}{(r_{new})^2} = G\frac{mM_e}{2R_e r_{new}} \quad \Rightarrow \quad \frac{(4)}{(r_{new})} = \frac{1}{2R_e} \quad \Rightarrow \quad r_{new} = 8R_e$$

This wouldn't happen, though, as the new motion would become elliptical looking something like:



Relationships--Simple Harmonic Motion

Relationships always true:

$\frac{d^2x}{dt^2} + (\kappa)x = 0$ or $\alpha + (\kappa)\theta = 0$	characteristic equation for simple harmonic motion
$\omega = (\kappa)^{1/2}$	angular frequency from characteristic equation
$\omega = 2\pi\nu$	angular frequency and frequency related
$T = \frac{1}{\nu}$	period inversely related to frequency
$x = A\cos(\omega t + \phi)$	position function for s.h.m.
$v = {dx}/{dt}$ and $a = {dv}/{dt} = {d^2x}/{dt^2}$	velocity and acceleration functions
$v_{max} = \omega A$	maximum velocity (happens at equilibrium)
$a_{max} = \omega^2 A$	maximum acceleration (happens at extremes)

Summary of Relationships

For a spring:

$$F_{sp} = -kx\hat{i}$$

for idea spring, Hooke's Law

$$\frac{d^2x}{dt^2} + \left(\frac{k}{m}\right)x = 0$$

characteristic equation for simple harmonic motion

$$\omega = \left(\frac{k}{m}\right)^{1/2}$$

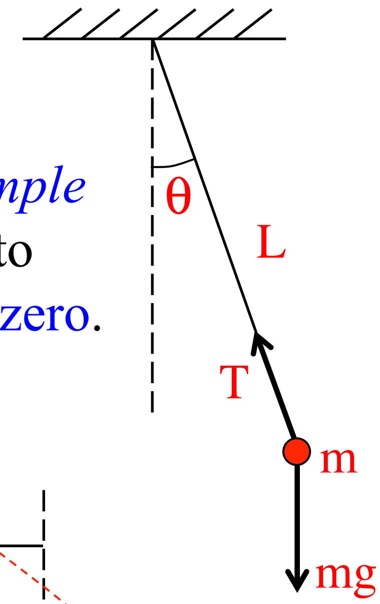
angular frequency from characteristic equation

$$E_{tot} = \frac{1}{2}kA^2$$

total mechanical energy in system

The period and frequency of oscillation for a spring is *constant* no matter what the spring's amplitude. How so? A larger displacement will require more distance traveled to execute a single cycle, but because force is a function of displacement, it will also generate a larger maximum force, so the *period* will stay the same no matter what!

Example 2: A simple pendulum of length *L* is observed as shown to the right. What is its period of motion?



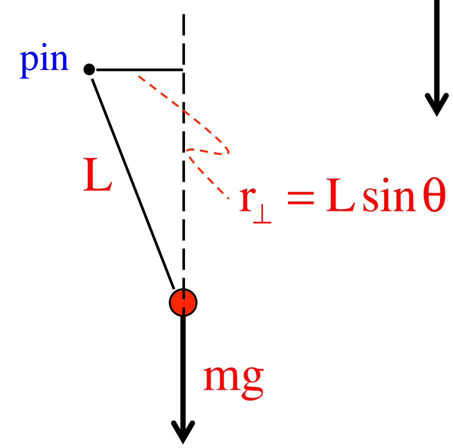
If we can show that this system's N.S.L. expression conforms to *simple harmonic motion*, we have it. As the motion is rotational, we need to sum torques about the pivot point. The torque due to the *tension* is zero. Noting that *r-perpendicular* for gravity is $L\sin\theta$, we can write:

$$\sum \tau_{pin}:$$

$$-(\cancel{m}g)(\cancel{L}\sin\theta) = I_{piin}\alpha$$

$$= (\cancel{m}L^{\cancel{2}})\frac{d^2\theta}{dt^2}$$

$$\Rightarrow \quad \frac{d^2\theta}{dt^2} + \left(\frac{g}{L}\right)\sin\theta = 0$$



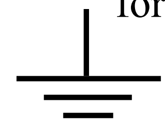
This isn't quite the right form, but if we take a *small angle approximation*, we find that for $\theta <<$, $\sin\theta \rightarrow \theta$ and we can write:

$$\frac{d^2\theta}{dt^2} + \left(\frac{g}{L}\right)\theta = 0$$

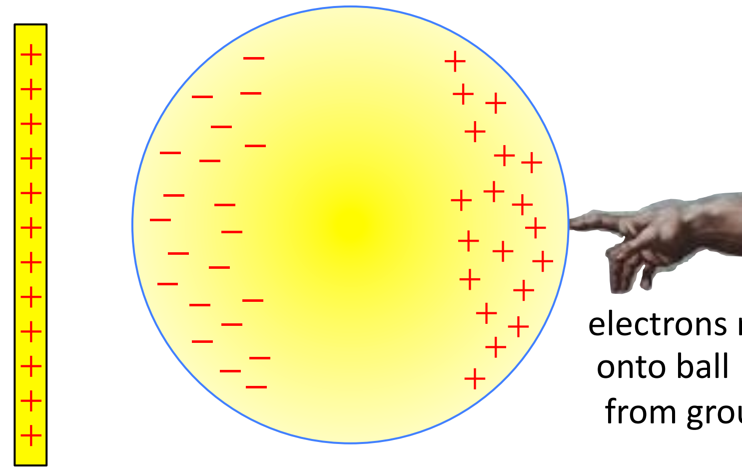
Electrostatics

Grounding is the connecting of a structure to a *reservoir of charge*, often quite literally the ground, to electrically neutralize the structure.

The symbol for ground in an electrical circuit is shown to the right.

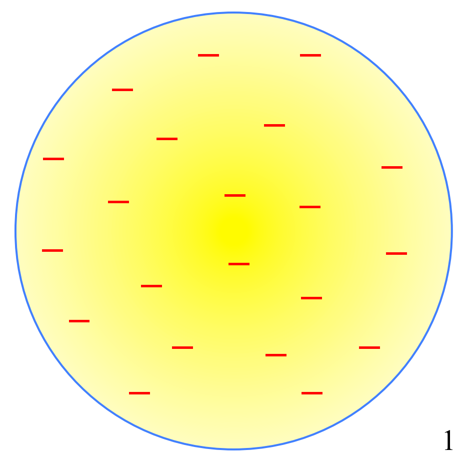


Touching our polarized ball on the side opposite the rod (I'd like to thank Michelangelo for the hand) will "ground" that side, allowing free electrons to run from the hand to the ball, neutralizing that side of the ball.

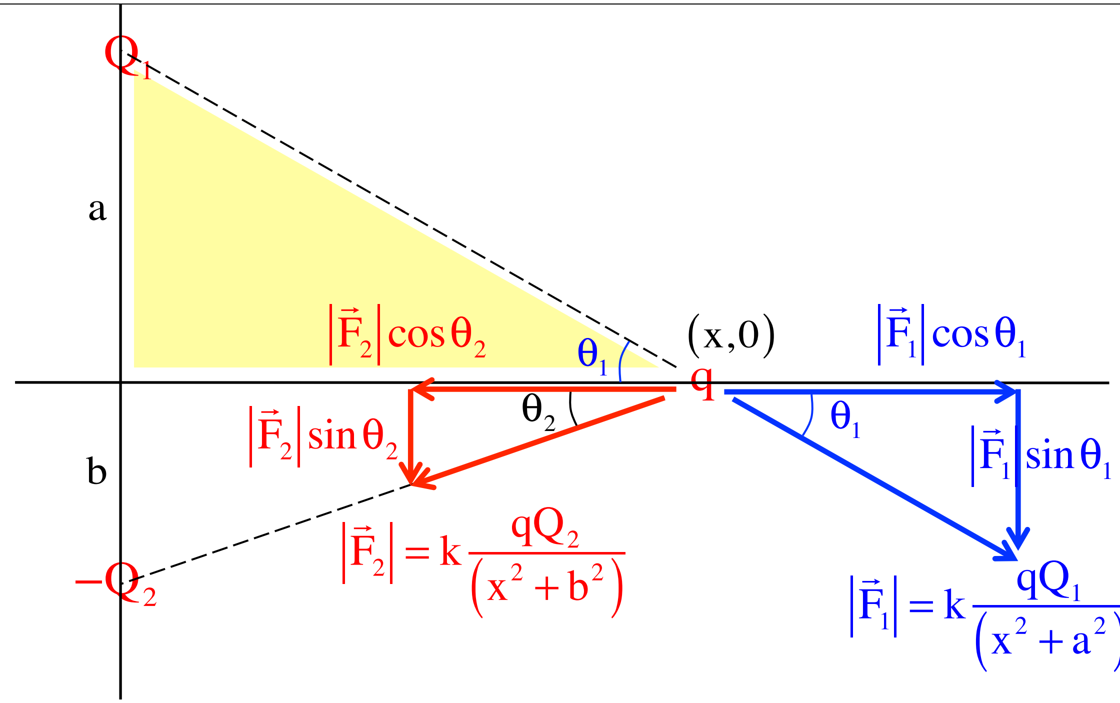


Remove the hand and rod, and the electrons will redistribute themselves leaving you with a *negatively charged* ball.

This is called CHARGING BY INDUCTION (you are inducing a charge separation, then removing charge by grounding).



Putting it all together:



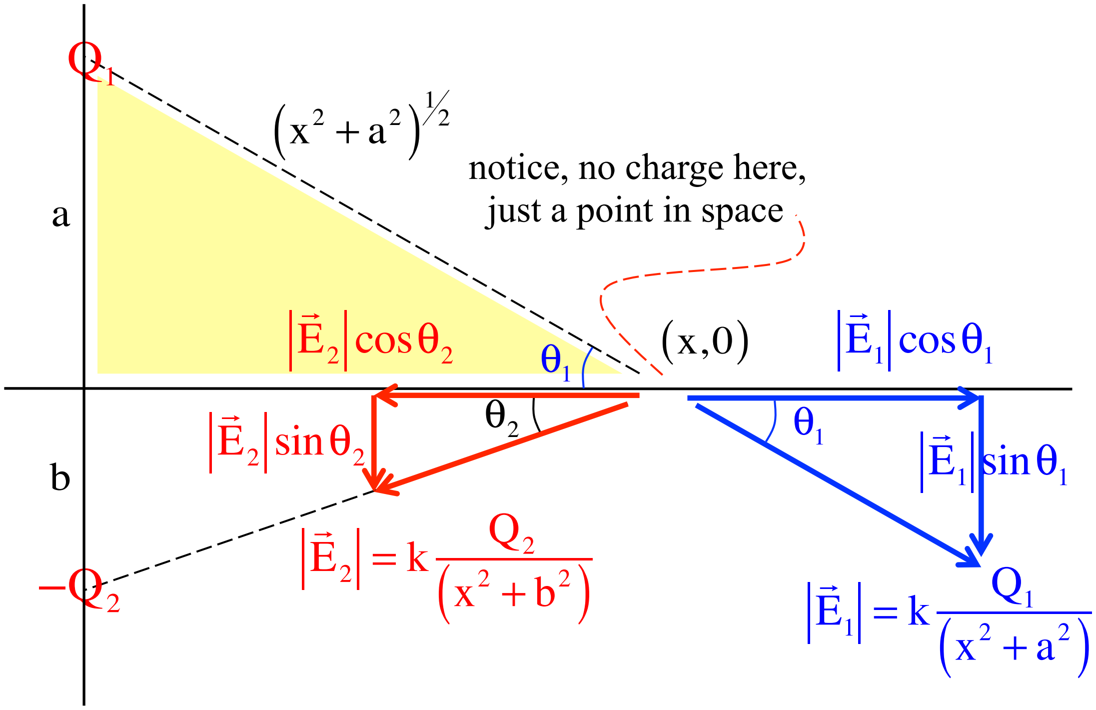
$$\vec{F}_C = \left(|F_1| \cos\theta_1 - |F_2| \cos\theta_2 \right)(\hat{i}) + \left(-|F_1|\sin\theta_1 - |F_2|\sin\theta_2\right)(\hat{j})$$

$$= \left(\frac{1}{4\pi\varepsilon_o}\frac{qQ_1}{(x^2+a^2)}\left(\frac{x}{(x^2+a^2)^{1/2}}\right) - \frac{1}{4\pi\varepsilon_o}\frac{qQ_2}{(x^2+b^2)}\left(\frac{x}{(x^2+b^2)^{1/2}}\right)\right)(\hat{i}) + \ \ldots$$

$$= \left(\frac{1}{4\pi\varepsilon_o}\frac{qQ_1 x}{(x^2+a^2)^{3/2}} - \frac{1}{4\pi\varepsilon_o}\frac{qQ_2 x}{(x^2+b^2)^{3/2}}\right)(\hat{i}) + \left(-\frac{1}{4\pi\varepsilon_o}\frac{qQ_1 a}{(x^2+a^2)^{3/2}} - \frac{1}{4\pi\varepsilon_o}\frac{qQ_2 b}{(x^2+b^2)^{3/2}}\right)(\hat{j})$$

Example 7: Derive an expression for the electric field at (x,0).

This is very similar to Example 3 (and because the charge q was positive in that problem, the forces on it and the direction of the electric fields will even be the same). The difference? No need to include the test charge q, just work with E:



Defining the field directions and magnitudes, then break into components:

$$\vec{E} = \left(|\vec{E}_1|\cos\theta_1 - |\vec{E}_2|\cos\theta_2\right)(\hat{i}) + \left(-|\vec{E}_1|\sin\theta_1 - |\vec{E}_2|\sin\theta_2\right)(\hat{j})$$

You'd use the same trickery ($\sin\theta_1 = \frac{a}{(x^2+a^2)^{1/2}}$) and finish the problem just like before.

An additional bit of trickery is involved in exploiting the symmetry of the set-up. Notice there is a second *dq* on the right side at an angle θ that will produce a mirror-image *differential electric field* to our original bit of charge. The *x-components* of the two fields will add to zero, so all we really have to deal with is the *y-component*.

With the *linear charge density* as:

$$\lambda = \frac{Q}{\left(2\pi R/2\right)} = \frac{Q}{\pi R}$$

and *E* as: $E = \frac{k\lambda}{R}d\theta$

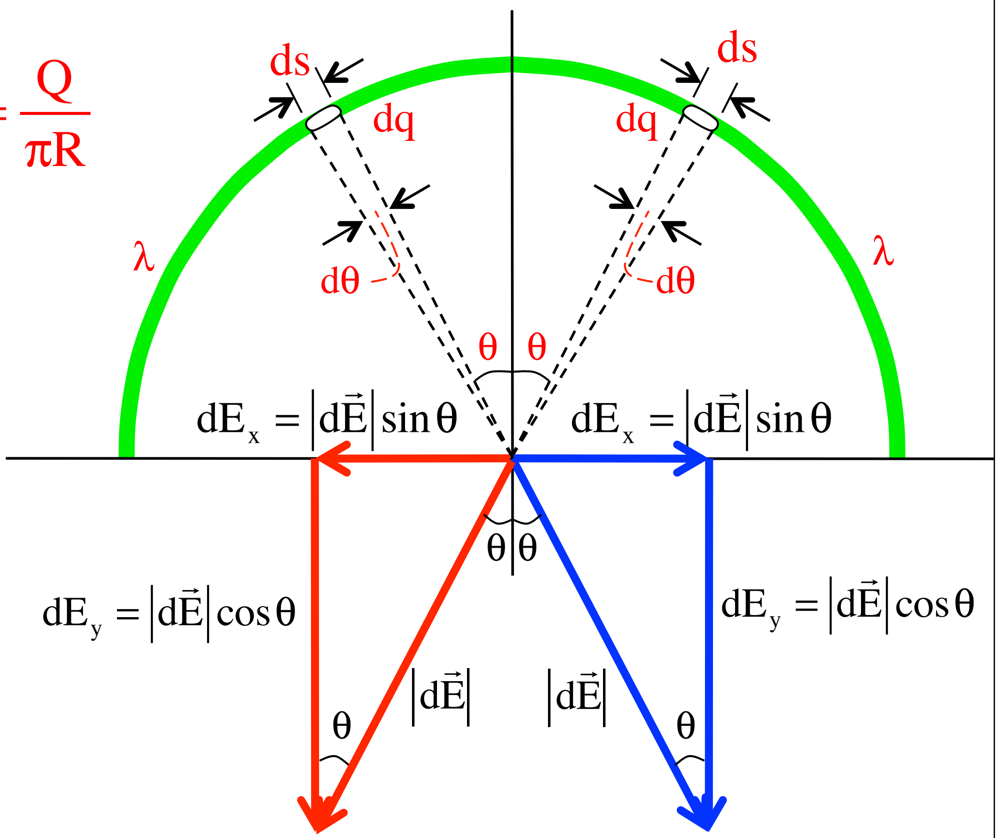
we can write:

$$\vec{E} = 2\int dE_y\left(-\hat{j}\right) = 2\int dE\cos\theta\left(-\hat{j}\right)$$

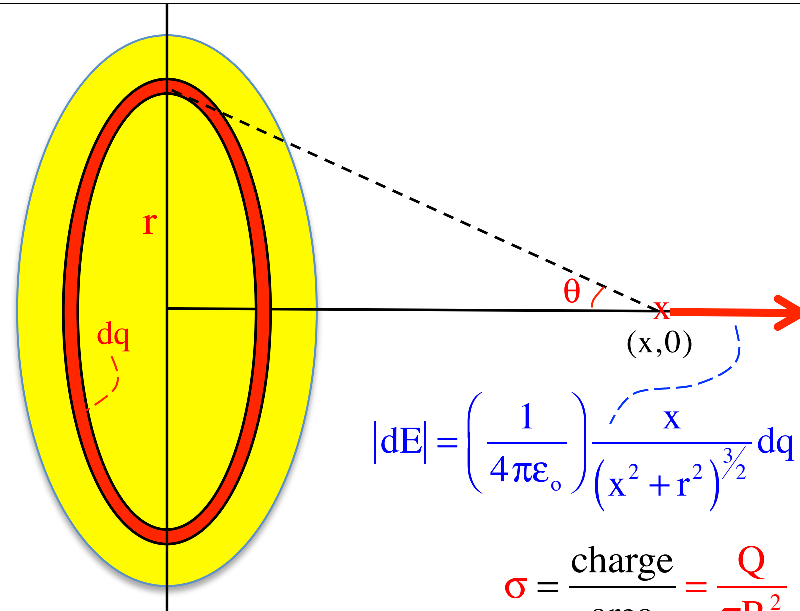
$$= \left[-2\left(\frac{1}{4\pi\varepsilon_o}\frac{\lambda}{R}\right)\int_{\theta=0}^{\pi/2}\cos\theta\, d\theta\right]\left(\hat{j}\right)$$

$$= \left[-2\left(\frac{1}{4\pi\varepsilon_o}\frac{\left(Q/\pi R\right)}{R}\right)\sin\theta\Big|_{\theta=0}^{\pi/2}\right]\left(\hat{j}\right)$$

$$= -\frac{Q}{2\pi^2\varepsilon_o R^2}\left(\hat{j}\right)$$



Summing over all of the differential hoops yields:



$$|dE| = \left(\frac{1}{4\pi\varepsilon_o}\right)\frac{x}{\left(x^2+r^2\right)^{3/2}}dq$$

$$\sigma = \frac{\text{charge}}{\text{area}} = \frac{Q}{\pi R^2}$$

$$dq = (2\pi\sigma r)dr$$

$$|E| = \left(\frac{1}{4\pi\varepsilon_o}\right)\frac{x}{\left(x^2+r^2\right)^{3/2}}dq$$

$$= \left(\frac{1}{4\pi\varepsilon_o}\right)\int_{r=0}^{R}\frac{x}{\left(x^2+r^2\right)^{3/2}}(2\pi\sigma r)dr$$

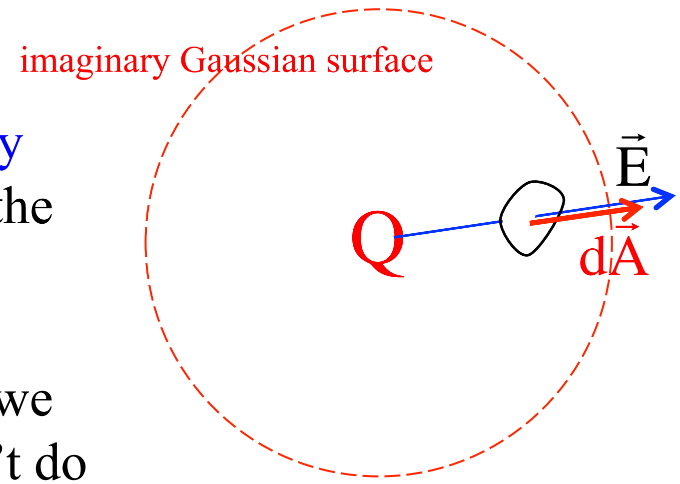
rewriting

$$= \left(\frac{2\pi x\sigma}{4\pi\varepsilon_o}\right)\int_{r=0}^{R}\frac{r}{\left(x^2+r^2\right)^{3/2}}dr = \left(\frac{2\pi x\sigma}{4\pi\varepsilon_o}\right)\int_{r=0}^{R} r\left(x^2+r^2\right)^{-3/2}dr$$

$$= \left(\frac{2\pi x\sigma}{4\pi\varepsilon_o}\right)(-1)\left(x^2+r^2\right)^{-1/2}\Big|_{r=0}^{R} = \left(\frac{\cancel{2}\cancel{\pi}x\left(\frac{Q}{\cancel{\pi}R^2}\right)}{\cancel{4}_{2}\pi\varepsilon_o}\right)(-1)\left(\frac{1}{\left(x^2+R^2\right)^{1/2}}-\frac{1}{\left(x^2\right)^{1/2}}\right)$$

$$= \left(\frac{Q}{2\pi R^2\varepsilon_o}\right)(-1)\left(\frac{x}{\left(x^2+R^2\right)^{1/2}}-\frac{x}{x}\right) = \left(\frac{Q}{2\pi R^2\varepsilon_o}\right)\left(1-\frac{x}{\left(x^2+R^2\right)^{1/2}}\right)$$

Gauss's Law



Herein lies the beauty of the method. Because every point on the surface is equidistant from the charge, the evaluation of the E at *every differential surface dA* WILL BE THE SAME, which is to say, IS A CONSTANT VALUE, and because it is a constant, we can pull it out of the integral. (Note that we couldn't do that with the original Example 1 because each point was a different distance from Q.) With that, we can write:

$$\int_S |\vec{E}||d\vec{A}| = \frac{Q}{\varepsilon_o}$$

$$\Rightarrow |\vec{E}| \int_S |d\vec{A}| = \frac{Q}{\varepsilon_o}$$

That makes life wonderful, as now the only thing inside the integral is the differential surface area *dA*, and summing that over the surface simply yields the total surface area of the sphere ($4\pi R^2$) . . . So we can further write

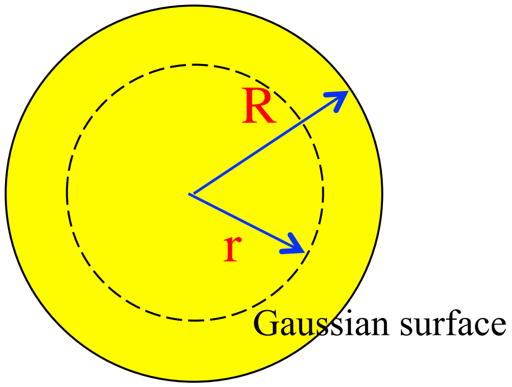
$$|\vec{E}| \int_S |d\vec{A}| = \frac{Q}{\varepsilon_o}$$

$$\Rightarrow |\vec{E}|\left(4\pi R^2\right) = \frac{Q}{\varepsilon_o}$$

$$\Rightarrow |\vec{E}| = \frac{Q}{4\pi\varepsilon_o R^2}$$

Look familiar? It should. It's the same as the *electric field function* we derived for a *point charge* using Coulomb's Law!

b.) for r<R: (con't.—doing this with the *density function* . . . though either way would do here)



$$\int_S \vec{E} \bullet d\vec{A} = \frac{\int_{a=0}^{r} \rho \, dV}{\varepsilon_o}$$

$$\Rightarrow \quad \int_S E\, dA \cos 0^o = \frac{\int_{a=0}^{r} \left[\left(\frac{-Q}{\left(\cancel{4}/3\,\cancel{\pi} R^3\right)} \right) \right] \left[\cancel{4}\cancel{\pi} a^2 da \right]}{\varepsilon_o}$$

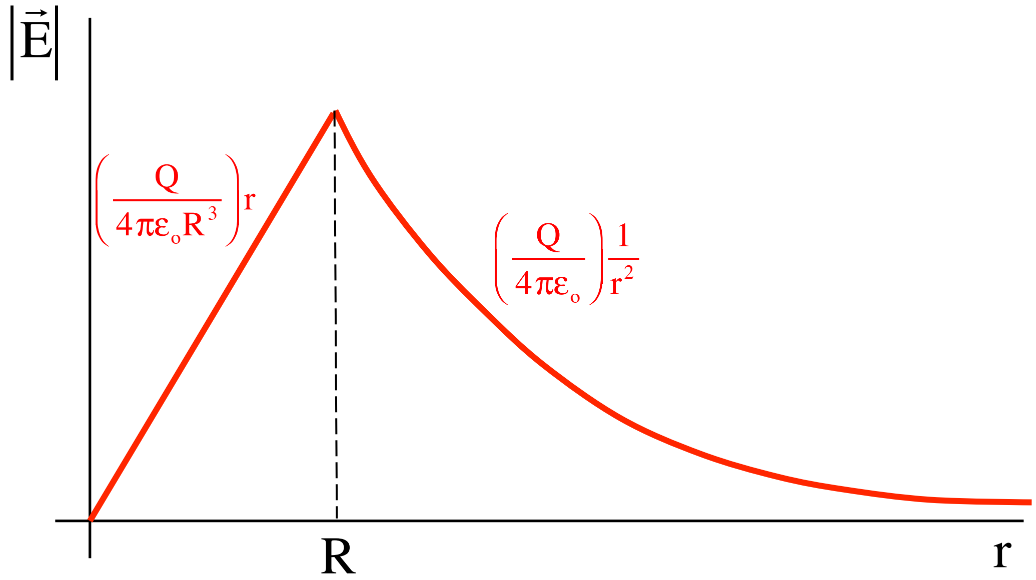
$$\Rightarrow \quad E \int_S dA = \frac{-\left(r^3 / R^3 \right) Q}{\varepsilon_o}$$

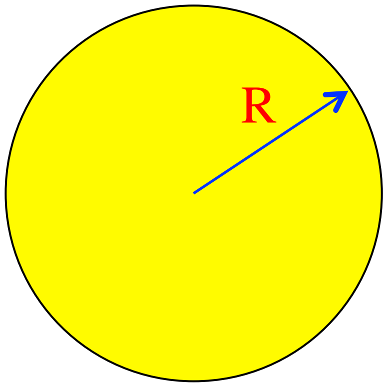
$$\Rightarrow \quad E\left(4\pi r^{\cancel{2}}\right) = \frac{-\left(r^{\cancel{3}} / R^3 \right) Q}{\varepsilon_o}$$

$$\Rightarrow \quad E = -\frac{Q}{4\pi\varepsilon_o R^3} r$$

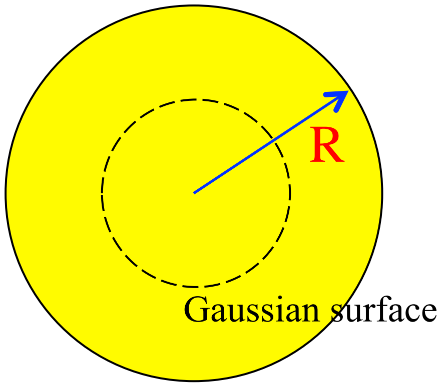
Note that this is a *linear E-field* inside the sphere!

c.) What does the graph look like for *electric field magnitude* versus *position*?





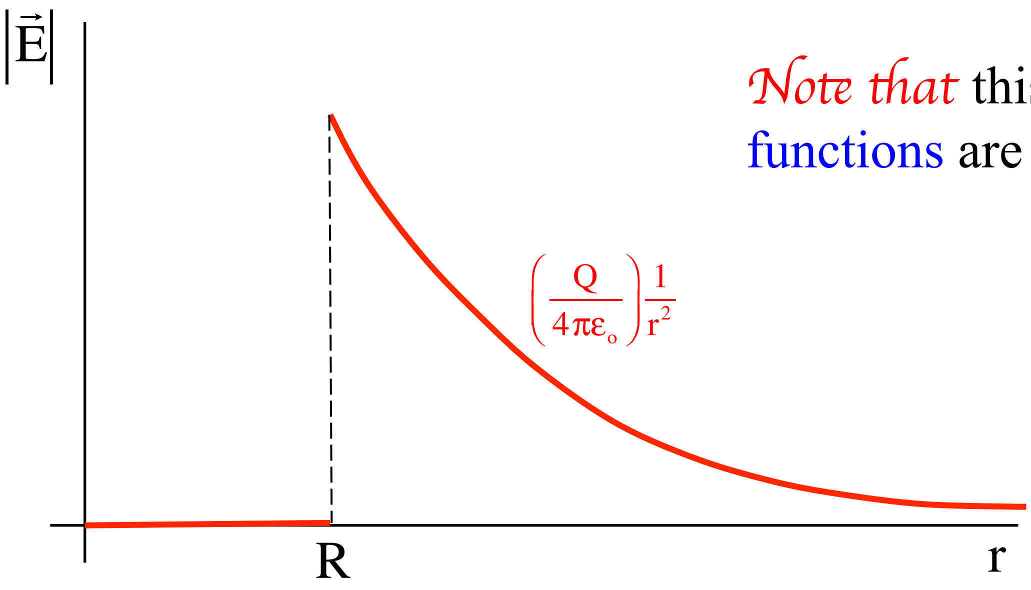
b.) for r<R: This is easy. With all the charge on the surface, the charge enclosed inside the Gaussian surface is zero and:



$$\int_S \vec{E} \bullet d\vec{A} = \frac{0}{\varepsilon_o}$$

$$\Rightarrow \; E = 0$$

c.) What does the graph look like for *E-field* magnitude versus *position*?



Note that this suggests that *E-fld* functions are *discontinuous* . . .

With that, Gauss's Law becomes:

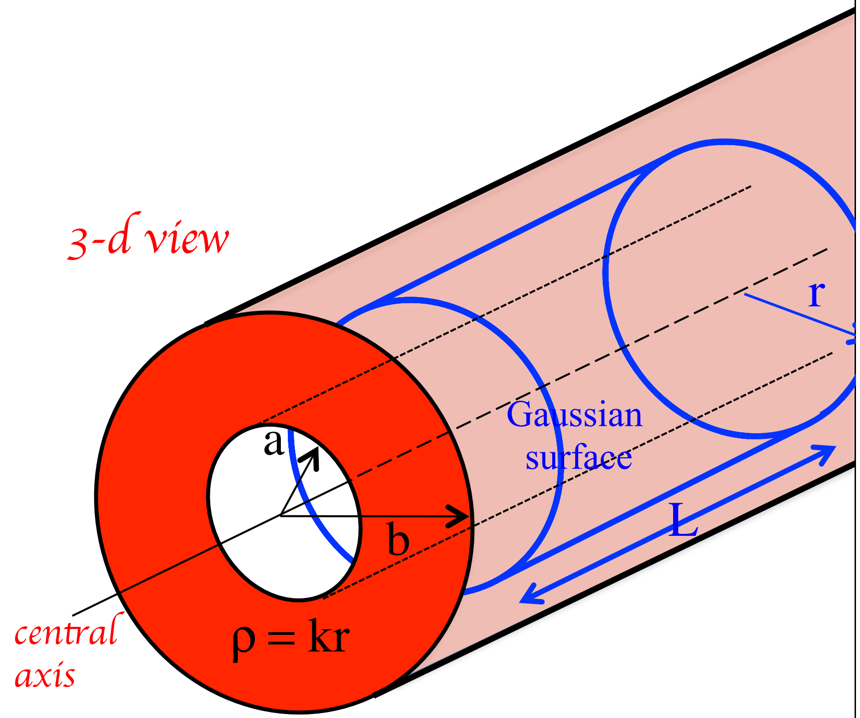
$$\int_S \vec{E} \bullet d\vec{S} = \frac{\int_{c=a}^{r} \rho \, dV}{\varepsilon_o}$$

$$\Rightarrow \quad E(2\pi r L) = \frac{\int_{c=a}^{r} (kc)\left[(2\pi c L)dc\right]}{\varepsilon_o}$$

$$\Rightarrow \quad E = \frac{\cancel{2}\cancel{\pi}k\cancel{L}}{\cancel{2}\cancel{\pi}\varepsilon_o r\cancel{L}} \int_{c=a}^{r} c^2 \, dc$$

$$= \frac{k}{\varepsilon_o r}\left(\frac{c^3}{3}\right)\Bigg|_{c=a}^{r}$$

$$= \frac{k}{3\varepsilon_o r}\left(r^3 - a^3\right)$$



b.) Derive an *electric field* for $r < a$: (It's zero as no charge inside Gaussian surface.)

c.) Derive an *electric field* for $r > b$:

Same problem as *Part a* exception of the limits of the integration are different (you are now adding up ALL the charge inside the cylinder, so the limits go from $c = a$ to $c = b$ instead of $c = a$ to $c =$ *the Gaussian radius* r.)

$$\int_{curve} \vec{E} \bullet d\vec{A}_{curve} + 2\int_{end} \vec{E} \bullet d\vec{A}_{end} = \frac{q_{cnclosed}}{\varepsilon_o}$$

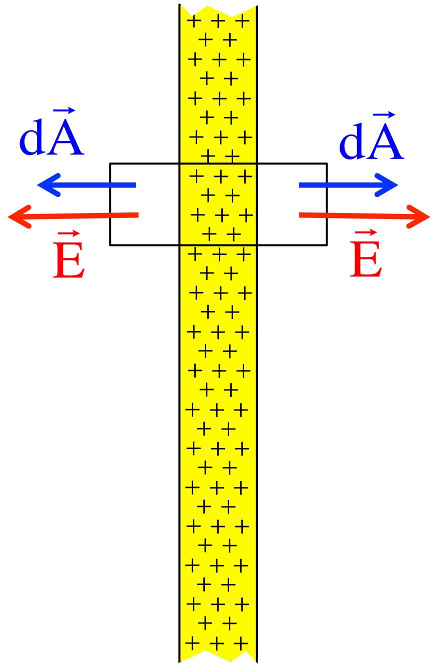
$$\Rightarrow \int_{curve} (E)\, dA_{curve} \underbrace{\cancel{\cos 90^o}}_{0} + 2\int_{end} (E)\, dA_{end} \cos 0^o = \frac{\sigma_{ins} A_{end}}{\varepsilon_o}$$

$$\Rightarrow 2E\int_{end} dA_{end} = \frac{\sigma_{ins} A_{end}}{\varepsilon_o}$$

$$\Rightarrow 2E\cancel{A}_{end} = \frac{\sigma_{ins} \cancel{A}_{end}}{\varepsilon_o}$$

$$\Rightarrow E = \frac{\sigma_{ins}}{2\varepsilon_o}$$

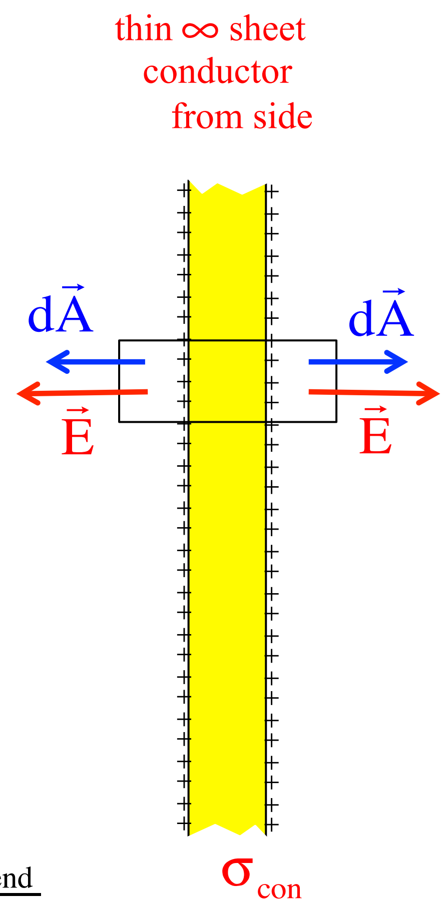
thin ∞ sheet
insulator
from side

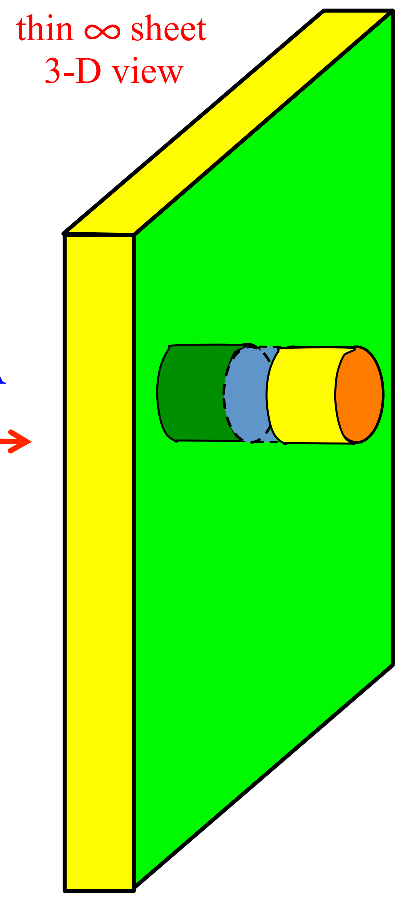


σ_{ins}

Example 11: Derive an expression for the *electric field function* for an "infinite" sheet of conducting material whose area charge density is a constant σ_{con}.

Here is where the difference in the charge configurations comes into play. We could use the same plug we used with the insulator, but there would be two surfaces upon which there was charge placed, each of which would have a charge density of σ_{con}. That means:





$$\cancelto{0}{\int_{curve} \vec{E} \bullet d\vec{A}_{curve}} + 2\int_{end} \vec{E} \bullet d\vec{A}_{end} = \frac{q_{enclosed}}{\varepsilon_o}$$

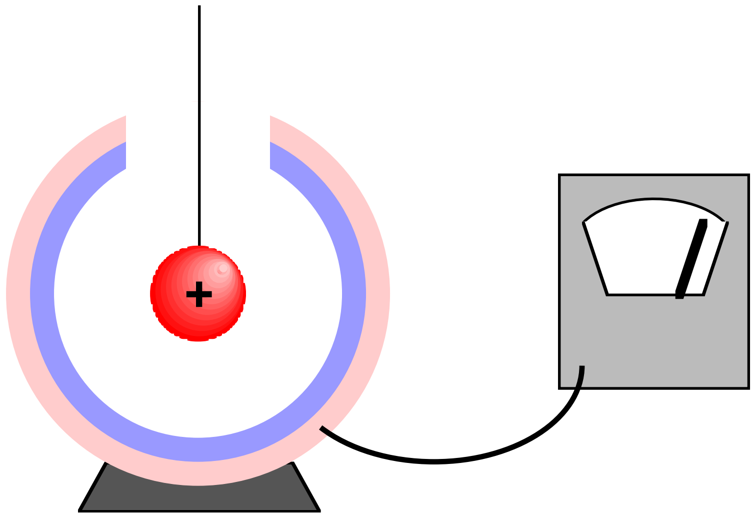
$$\Rightarrow \quad 2E\int_{end} dA_{end} = \frac{\sigma_{con}A_{end} + \sigma_{con}A_{end}}{\varepsilon_o}$$

$$\Rightarrow \quad 2E\cancel{A}_{end} = \frac{2\sigma_{con}\cancel{A}_{end}}{\varepsilon_o}$$

$$\Rightarrow \quad E = \frac{\sigma_{con}}{\varepsilon_o}$$

Example 16 - Faraday's Ice Pail Experiment

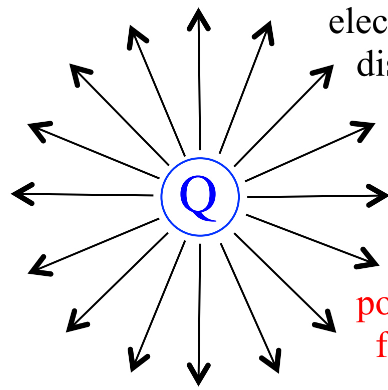
Courtesy of Mr. White:



Electric Potentials

(energy considerations)

An ELECTRICAL POTENTIAL FIELD, measuring the amount of *potential energy per unit charge AVAILABLE* at all points in the region of a *field-producing* charge, can be (and is) associated with *any* charge configuration.



An ELECTRICAL POTENTIAL FIELD exists wherever there is charge (and, for that matter, wherever there is an *electric field*). For the potential fields to exist, there doesn't need to be present a *secondary charge* to feel the effect. And because *voltage-flds* tell us how much energy is available *PER UNIT CHARGE* at a point, the *electrical potential field V* is defined as:

$$V = U/q$$

Example 1: How much *potential energy* does a 2 C charge have at a point where the *absolute electrical potential* is $V_1 = 3$ joules/coulomb?

$$V_1 = \frac{U_1}{q} \quad \Rightarrow \quad U_1 = qV_1$$
$$= (2\ \mathrm{C})(3\ \mathrm{J/C}) = 6\ \mathrm{J}$$

Work and Electrical-Potential (Voltage) Fields

Note: An *absolute electrical potential field* is a modified *potential energy* field.

Everything you can do with *energy considerations*, you can do with *electrical potential functions*:

Just as the work done on a body moving from one point to another in a *conservative force field* equals $W = -\Delta U$, we can use the definition of absolute electrical potential to write:

$$W = -\Delta U = -q\Delta V$$

$$\Rightarrow \frac{W}{q} = -\Delta V$$

and

Apparently, if you know the *voltage difference* between two points, you know how much *work per unit charge* AND *potential energy per unit charge* *t*he field has available between the two points.

$$W = -\Delta U = -q\Delta V$$

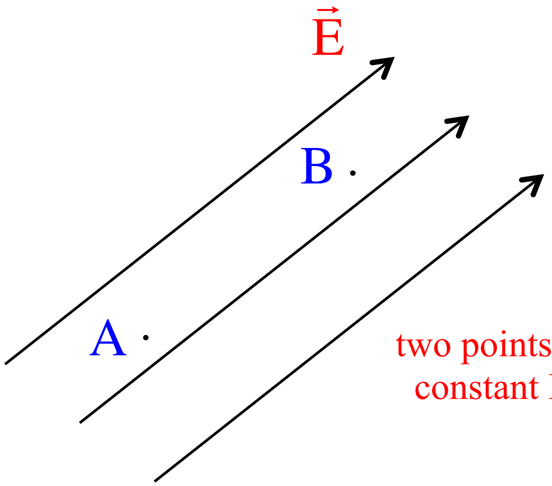
$$\Rightarrow \frac{\Delta U}{q} = \Delta V$$

Example 5: How much work does a field do on a moving 2 C charge if the *potential difference* between its beginning and end points is 7 volts?

$$\frac{W}{q} = -\Delta V \quad \Rightarrow \quad W = -q\Delta V$$

$$= -(2\ \text{C})(7\ \text{J/C}) = -14\ \text{J}$$

Electrical Potential Difference and E-flds



two points in a constant E-fld

Assuming we are dealing with a *constant electric field* and a *straight-line path* between two points in the field, we can use the definition of work ($W = \vec{F} \bullet \vec{d}$) with the manipulated definition of the electric field ($\vec{F} = q\vec{E}$) to extend out *potential difference relationship* ($W/q = -\Delta V$) into a very interesting proposition. Specifically:

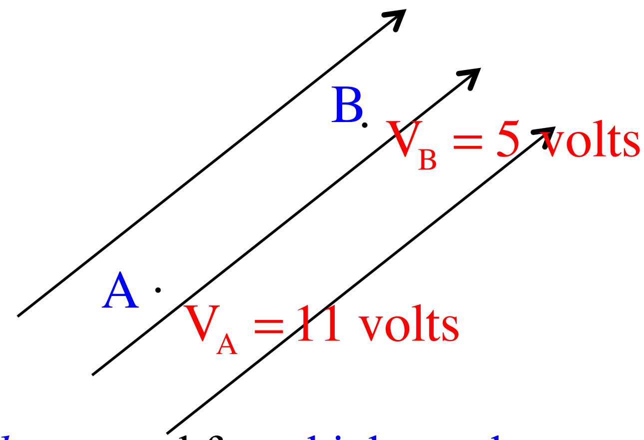
$$\frac{W_{AB}}{q} = -\Delta V_{AB} \quad \Rightarrow \quad \frac{\vec{F} \bullet \vec{d}_{AB}}{q} = -\Delta V_{AB}$$

$$\Rightarrow \quad \frac{\cancel{q}\vec{E} \bullet \vec{d}_{AB}}{\cancel{q}} = -\Delta V_{AB}$$

$$\Rightarrow \quad \vec{E} \bullet \vec{d}_{AB} = -\Delta V_{AB}$$

And what might we glean from this bit of amusement?

c.) A positive charge Q=1C and mass *m=1 kg* moves naturally along the *E-fld* lines.



i.) Is the charge moving from *higher electrical potential to lower*, or *lower electrical potential to higher*?

This has nothing to do with the charge. *Electric fields* proceed from higher voltage to lower, so it's doing the former.

ii.) Is the charge moving from higher *potential energy* to lower, or lower *potential energy* to higher?

This has EVERYTHING to do with the charge. POSITIVE CHARGES naturally move from *higher to lower voltage* along *E-fld* lines (being by definition the direction a positive charge would naturally accelerate), so it is moving from *higher* to lower potential energy.

iii.) If Q's initial velocity was 3 m/s at A, what is its velocity at B? (Note that the voltages have been put on the sketch.)

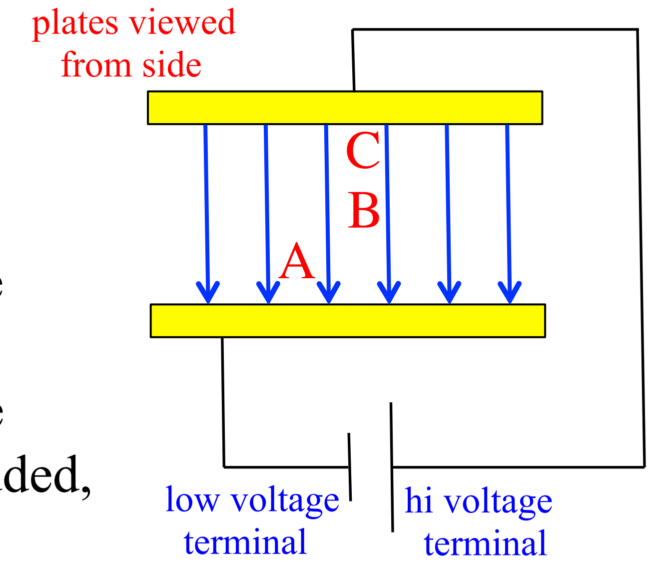
$$\sum KE_1 + \sum U_1 + \sum W_{ext} = \sum KE_2 + \sum U_2$$

$$\tfrac{1}{2} m v_A^{\ 2} + (qV_A) + 0 = \tfrac{1}{2} m v_B^{\ 2} + (qV_B)$$

$$\tfrac{1}{2}(1)(3)^2 + (1)(11) = \tfrac{1}{2}(1) v_B^{\ 2} + (1)(5)$$

$$\Rightarrow \quad v_B = 4.58 \text{ m/s}$$

e.) An electron $(e = 1.6x10^{-19} C,\ m = 9.1x10^{31} kg)$ accelerates between the plates. How fast is it moving if it started from rest?



Note that the electron (charge –e) would accelerate from the negative the positive plate, and that the *potential energy* of a charge sitting at a point whose potential is V is $U = qV$ with the charge's sign included, we can write:

$$\sum KE_1 + \sum U_1 + \sum W_{ext} = \sum KE_2 + \sum U_2$$

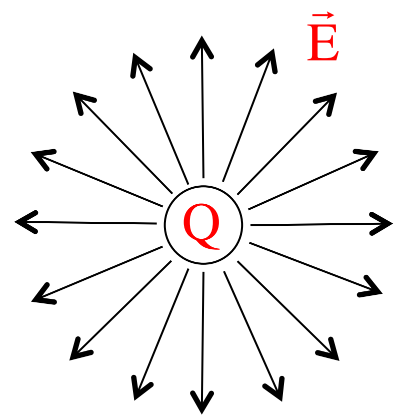
$$0 + \left((-e)V_-\right) + 0 = \tfrac{1}{2} mv^2 + \left((-e)V_+\right)$$

$$\Rightarrow \left(-1.6x10^{-19} C\right)(2\ V) = \tfrac{1}{2}\left(9.1x10^{-31}\ kg\right) v_B{}^2 + \left(-1.6x10^{-19} C\right)(14\ V)$$

$$\Rightarrow\ v = 2.1x10^6\ m/s$$

A Specific Case--The Electrical Potential Generated by a POINT CHARGE

Example 11: Derive a general expression for the *electrical potential* generated by a point charge *Q*?



Setting the zero point for the *electrical potential* to be where the *electric field* is zero (i.e., at infinity), and using the *electric field function* for a point charge as $\vec{E} = k\, {}^{Q}\!/_{r^2}\, \hat{r}$, we can write:

$$V(r) - V(\infty) = -\int_{\infty}^{r} \vec{E} \bullet d\vec{r}$$

$$= -\int_{r=\infty}^{r} \left(k\frac{Q}{r^2}\hat{r} \right) \bullet d\vec{r}$$

$$= -\int_{r=\infty}^{r} \left(k\frac{Q}{r^2} \right) dr\left(\cos 0^{o}\right)$$

$$= -kQ\left(-\frac{1}{r} \right)\Bigg|_{r=\infty}^{r}$$

$$\Rightarrow \quad V(r)_{pt\ chg} = \left(\frac{1}{4\pi\varepsilon_o} \right)\frac{Q}{r}$$

So How Are Electric Fields and Electrical Potentials Related?

Remember back to the Energy chapter when we related a conservative force function to its potential energy function. We found that the *spatial rate of change of potential energy* equals the *force* associated with the potential energy field, or $\vec{F} = -\left({dU}/{dx}\right)\hat{i}$. There is an electrical analogue to this.

That is, the differential consequence of:

$$V(r) - V(\text{zero pt}) = -\int_{\text{zero pt}}^{r} \vec{E} \bullet d\vec{r}$$

is

$$dV = -\vec{E} \bullet d\vec{r}$$

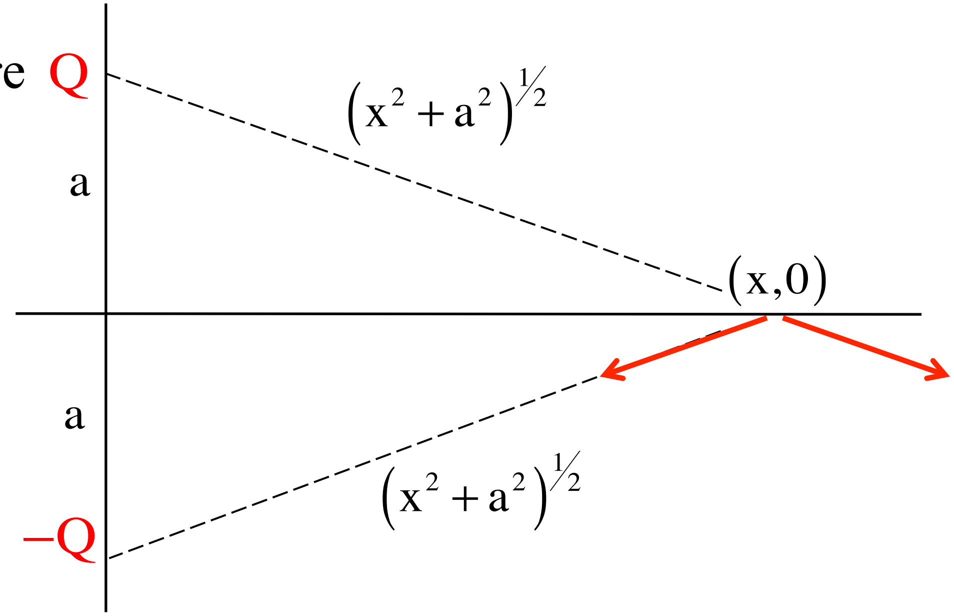
But if that is true, it must also be true that: $\vec{E} = -\dfrac{dV}{dr}\hat{r}$

Except in *Cartesian coordinates* (assuming *E* is in the x-direction), $\vec{E} = -\dfrac{dV}{dx}\hat{i}$

which can be expanded into multiple dimensions using the *del operator* as:

$$\vec{E} = -\vec{\nabla}V = -\left(\frac{\partial V}{\partial x}\hat{i} + \frac{\partial V}{\partial y}\hat{j} + \frac{\partial V}{\partial z}\hat{k}\right)$$

Example 12: Assume the charges are equal and opposite, and are placed symmetrically as shown.



a.) Is there an *electric field* at (x,0). If so, in what direction is it?

There will be an *E-fld* at (x,0). By inspection, its x-components will add to zero leaving it with only y-components.

b.) Is there an *absolute electrical potential* at (x,0). If so, in what direction is it?

TRICK QUESTION—*electrical potentials* don't have directions as they are scalars. As for magnitude:

$$V_{total} = V_Q + V_{-Q}$$

$$= \left(\frac{1}{4\pi\varepsilon_o}\right)\frac{Q}{\left(x^2+a^2\right)^{1/2}} + \left(\frac{1}{4\pi\varepsilon_o}\right)\frac{-Q}{\left(x^2+a^2\right)^{1/2}}$$

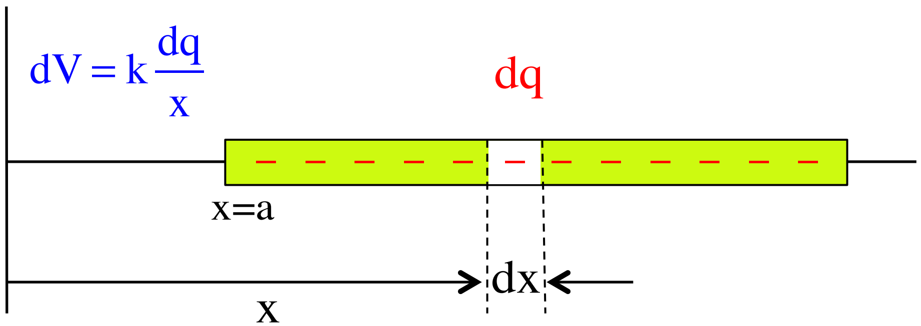
$$= 0$$

c.) Does this make sense?

Yes, if you understand how *E-flds* and *voltage flds* are related to one another.

Example 13 (A non-AP problem):

Derive an expression for the electrical potential at the origin due to a rod with charge *-Q* uniformly distributed over its length *L*.



This extended charge distribution is something you've already seen. The solving technique is exactly as was before. Define the differential electrical potential at the origin due to a differential bit of charge, then sum that differential electrical potential over the entire rod. You'll again need to define a *linear charge density* function $\lambda = -Q/L$ and note that $dq = \lambda dx$. With that, we can write:

$$V = \int dV = \int_{x=a}^{a+L} \frac{1}{4\pi\varepsilon_o}\frac{dq}{x}$$

$$= \int_{x=a}^{a+L} \frac{1}{4\pi\varepsilon_o}\frac{(\lambda dx)}{x} = \frac{\left(-Q/L\right)}{4\pi\varepsilon_o}\int_{x=a}^{a+L}\frac{dx}{x}$$

$$= \frac{\left(-Q/L\right)}{4\pi\varepsilon_o}\left(-\ln x\Big|_{x=a}^{a+L}\right) = \frac{-Q}{4\pi\varepsilon_o L}\left[\left(-\ln(a+L)\right) - \left(-\ln a\right)\right]$$

$$= \frac{-Q}{4\pi\varepsilon_o L}\left[\left(\ln(a) - \ln(a+L)\right)\right] = \frac{-Q}{4\pi\varepsilon_o L}\ln\left(\frac{a}{(a+L)}\right)$$

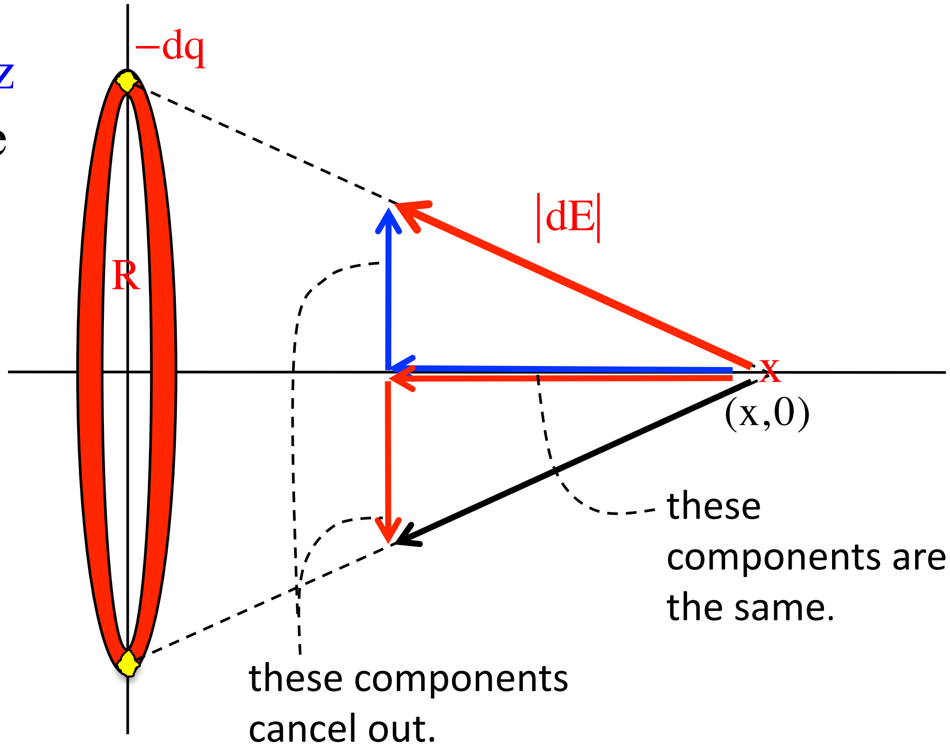
Example 15: A ring situated in the x-z plane (as shown) has *–Q's* worth of charge on it.

a.) What is the direction of the *E-fld* at (x,0)?

From observation, it's $-\hat{i}$.

b.) Derive an expression or *V* at (x,0)?

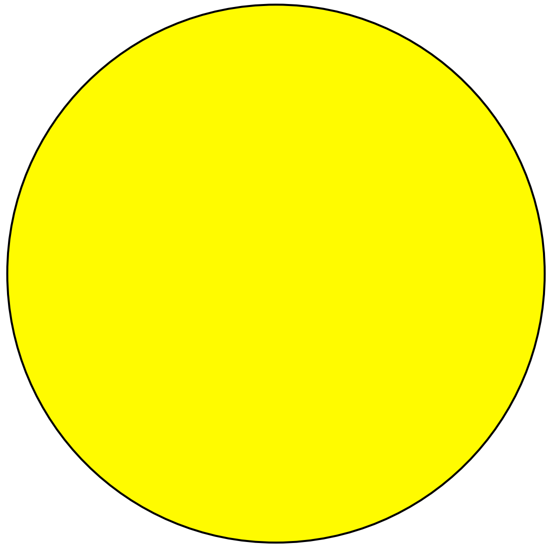
$$
\begin{aligned}
V &= \int dV \\
&= \frac{1}{4\pi\varepsilon_o}\int \frac{dq}{\left(x^2+R^2\right)^{1/2}} \\
&= \frac{1}{4\pi\varepsilon_o\left(x^2+R^2\right)^{1/2}}\int dq \\
&= \frac{-Q}{4\pi\varepsilon_o\left(x^2+R^2\right)^{1/2}}
\end{aligned}
$$

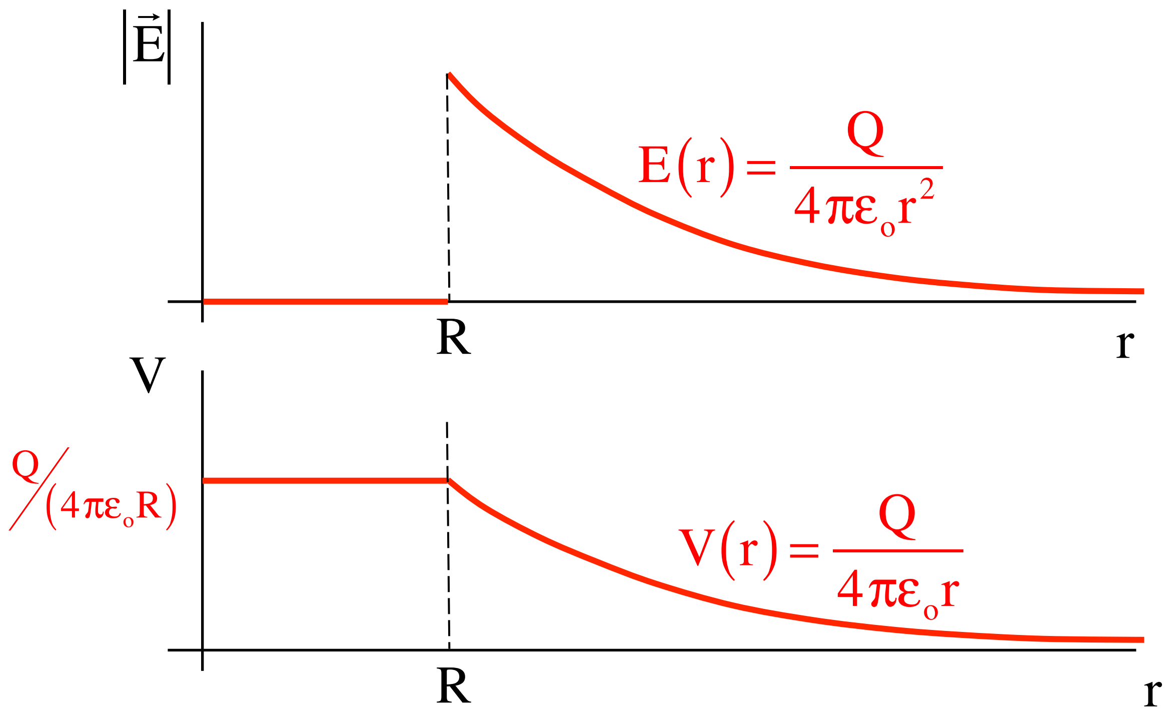


c.) Do the results from Parts a and b make sense together?

sure

c.) Sketch the graph for: *E-fld* vs *position* AND the *electrical potential field* vs. *position*.





Notice that whereas the *E-fld* functions is *discontinuous*, the *V-fld function* is *CONTINUOUS*!

SUMMARY—Conductors . . .

Electric Fields:

a.) Free charge on a conductor in a static setting stays on the conductor's surface.

b.) Close to the surface of a conductor, the *E-fld* is *perpendicular to the surface and* has a magnitude $E = \sigma / \varepsilon_o$.

c.) Inside a conductor, the *E-fld* is zero in a static charge situation (otherwise, electrons would migrate).

Electric Potentials:

a.) Free charge on a conductor will distribute itself so as to create a equipotential surface (the voltage will be the same at every point on the surface)..

b.) As the electric field inside a conductor is zero, the *voltage field* (the electrical potential field) inside a conductor will be CONSTANT.

Capacitance

Furthermore, the charge Q on ONE PLATE will always be proportional to the magnitude of the *voltage difference* across the plates, with the proportionality constant being the cap's *capacitance*. Mathematically, then:

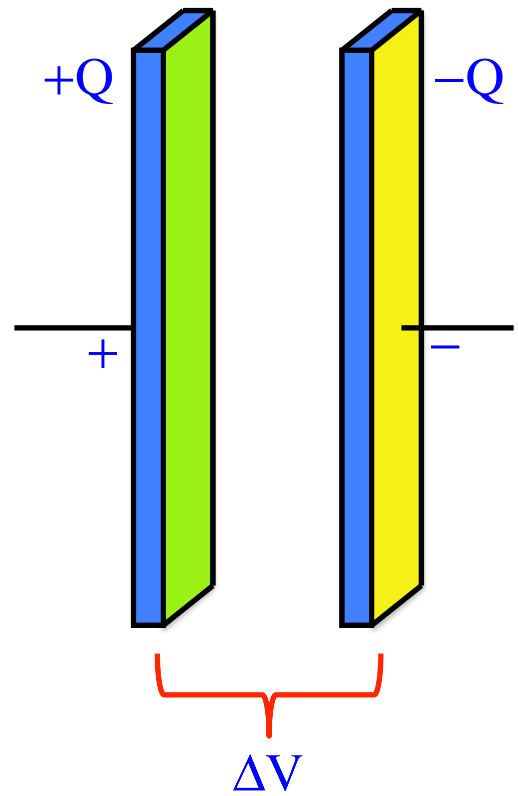
$$Q_{\text{on one plate}} = C(\Delta V)_{\text{across plates}}$$

Usually written in truncated form as:

$$Q = CV$$

this also means that the capacitance is defined as:

$$C = \frac{Q}{V}$$

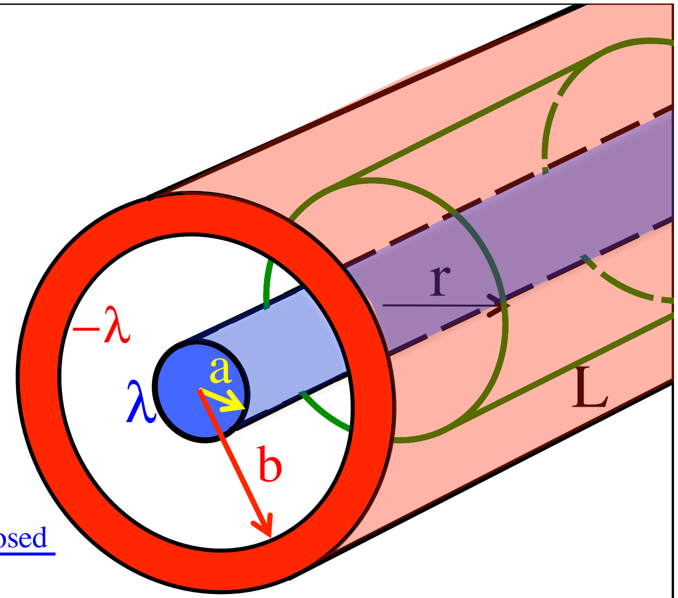


This, in turn, means the *capacitance* of a capacitor is a constant that tells you how much *charge per volt* the capacitor has the capacity to hold.

Its unit of coulombs per volt is given a special name—the farad.

It's not uncommon to find capacitors in the range of: millifarad ($\text{mf} = 10^{-3}\,\text{f}$), or microfarad ($\text{Mf}$ or $\mu\text{f} = 10^{-6}\,\text{f}$), or nanofarad ($\text{nf} = 10^{-9}\,\text{f}$), or picofarad ($\text{pf} = 10^{-12}\,\text{f}$).

Example 5: Derive an expression for the capacitance-per-unit-length of a coaxial cable of inside radius *a* and outside radius *b*.



1.) *Assume charges* (in this case, a linear charge density λ):

2.) *Noting that* all the charge will migrate to the inside surfaces, use a Gaussian cylinder of length *L* and *Gauss's Law* to derive an expression for the *E-fld* between plates.

$$\int_S \vec{E} \cdot d\vec{S} = \frac{q_{enclosed}}{\varepsilon_o}$$

$$\Rightarrow \quad \left|\vec{E}\right|(2\pi r L) = \frac{\lambda L}{\varepsilon_o}$$

$$\Rightarrow \quad E = \frac{\lambda}{2\pi\varepsilon_o r}$$

3.) *Derive an expression* for the *electrical potential difference* (V_{cap}) between the plates:

$$V_{cap} = -\Delta V = +\int \vec{E} \cdot d\vec{r}$$

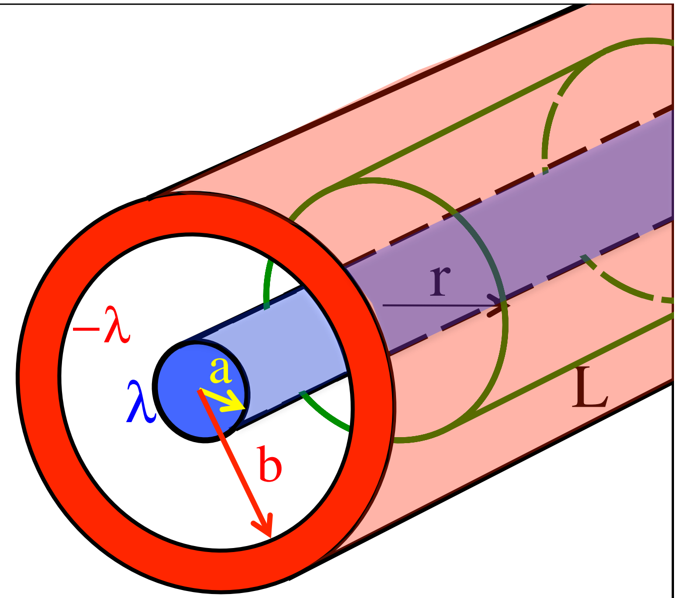
$$= \int_{r=a}^{b} \left(\frac{\lambda}{2\pi\varepsilon_o r} \hat{r} \right) \cdot (dr\ \hat{r}) = \frac{\lambda}{2\pi\varepsilon_o} \int_{r=a}^{b} \frac{1}{r} dr \cancelto{1}{\cos 0^o}$$

$$= \frac{\lambda}{2\pi\varepsilon_o} \ln(r) \Big|_{r=a}^{b} = \frac{\lambda}{2\pi\varepsilon_o} \left[\ln(b) - \ln(a) \right] = \frac{\lambda}{2\pi\varepsilon_o} \ln\left(\frac{b}{a}\right)$$

Except $\lambda = \frac{Q}{L}$

so

$$V_{cap} = \frac{\lambda}{2\pi\varepsilon_o}\ln\left(\frac{b}{a}\right)$$
$$= \frac{\left(Q/L\right)}{2\pi\varepsilon_o}\ln\left(\frac{b}{a}\right)$$

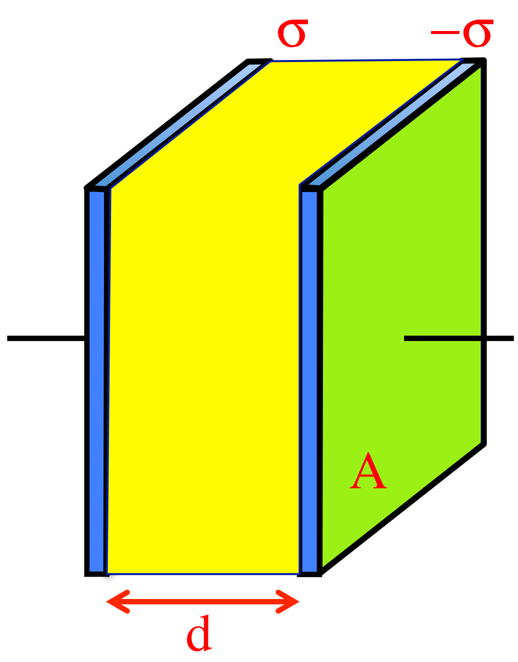


4.) Using the definition of capacitance:

$$C_{\text{parallel plate cap}} = \frac{\left(Q_{\text{on one plate}}\right)}{\left(V_{\text{across plates}}\right)}$$
$$= \frac{\cancel{Q}}{\frac{\left(\cancel{Q}/L\right)}{2\pi\varepsilon_o}\ln\left(\frac{b}{a}\right)}$$
$$\Rightarrow \quad C/L = \frac{2\pi\varepsilon_o}{\ln\left(\frac{b}{a}\right)}$$

So the parallel-plate capacitor derivation would look like:

$$\int_S \vec{E}\bullet d\vec{S} = \frac{q}{\kappa\varepsilon_o} \Rightarrow E\cancel{A} = \frac{\sigma\cancel{A}}{\kappa\varepsilon_o} \Rightarrow E = \frac{\sigma}{\kappa\varepsilon_o}$$



with $\sigma = \frac{Q}{A}$,

$$E = \frac{\sigma}{\kappa\varepsilon_o} = \frac{\left(Q/A\right)}{\kappa\varepsilon_o} = \frac{Q}{\kappa\varepsilon_o A}$$

That means: $V_{cap} = -\Delta V = +\int \vec{E}\bullet d\vec{r}$

$$= \int_{r=0}^{d}\left(\frac{Q}{\kappa\varepsilon_o A}\hat{r}\right)\bullet\left(dr\ \hat{r}\right) = \frac{Q}{\kappa\varepsilon_o A}\int_{r=0}^{d} dr\cos 0^o$$

$$= \frac{Q}{\kappa\varepsilon_o A}\left(r\right)\Big|_{r=0}^{d} = \frac{Q}{\kappa\varepsilon_o A}d$$

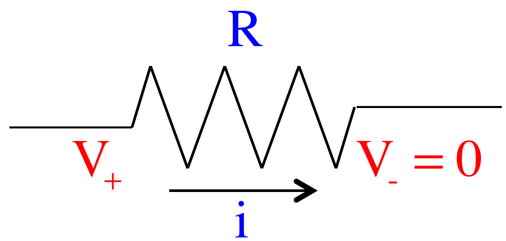
and

$$C_{\text{parallel plate cap}} = \frac{\left(Q_{\text{on one plate}}\right)}{\left(V_{\text{across plates}}\right)} = \frac{\cancel{Q}}{\left(\frac{\cancel{Q}}{\kappa\varepsilon_o A}d\right)}$$

$$= \kappa\varepsilon_o\frac{A}{d} \qquad \left(= \kappa C_{\text{w/o diel}}\right)$$

DC Circuits

Example 3: Current passes through a resistor? During some period of time, assume *q's* worth of charge passes through the resistor. How much work is done on that charge?



Assume the voltage on either side of the resistor is V_+ and $V_- = 0$ respectively. With that, we can write:

$$\begin{aligned} W = -\Delta U &= -q\Delta V \\ &= -q\left(\cancelto{0}{V_-} - V_+\right) \\ &= qV_+ \\ &= qV_R \end{aligned}$$

Example 4: How much power is being dissipated by the resistor in the previous problem?

$$P = \frac{W}{\Delta t} = \frac{-\Delta U}{\Delta t}$$

$$= \frac{qV_R}{\Delta t} = \left(\frac{q}{\Delta t}\right)V_R$$

$$= \quad i \quad V_R$$

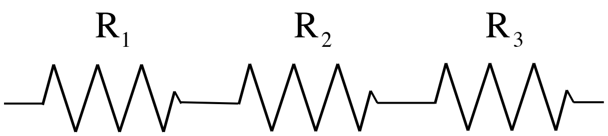
Power is work per unit time, so:

In short: $P = iV_R$

This is generally true, but according to Ohm's Law, we an write:

$$\begin{aligned} P = iV_R &= i(iR) \\ &= i^2R \end{aligned}$$

Characteristics of a Series Combinations



--*Each element* in a series combination is attached to its neighbor in *one place only*.

--*Current* is common to each element in a series combination.

--*There are* no nodes (junctions—places where current can slit up) internal to series combinations.

--*The equivalent resistance* for a series combination is: $R_{eq} = R_1 + R_2 + R_3 + ...$

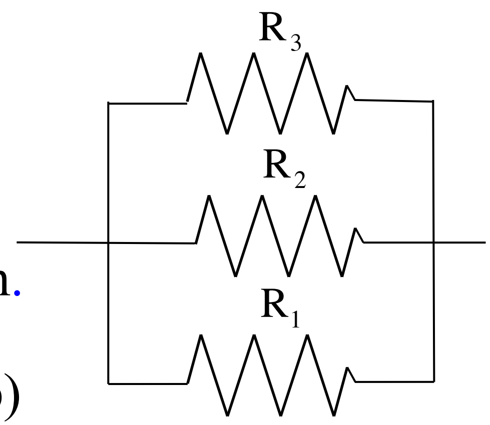
--*This means* the equivalent resistance is larger than the largest resistor in the combination;

--*This means* that if you add a resistor to the combination, R_{eq} will *increase* and the current through the combination (for a given voltage) will *decrease*.

Example 3: What's the equivalent resistance of a 5 Ω, 6 Ω and 7 Ω resistor in series?

$$R_{eq} = (5\Omega) + (6\Omega) + (7\Omega)$$
$$= 18\ \Omega$$

Characteristics of a Parallel Combinations



--Each element in a series combination is attached to its neighbor in *two place*.

--Voltage is common to each element in a parallel combination.

--There are nodes (junctions—places where current can slit up) internal to parallel combinations.

--The equivalent resistance for a parallel combination is: $\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \ldots$

--This means the equivalent resistance is SMALLER than the smallest resistor in the combination;

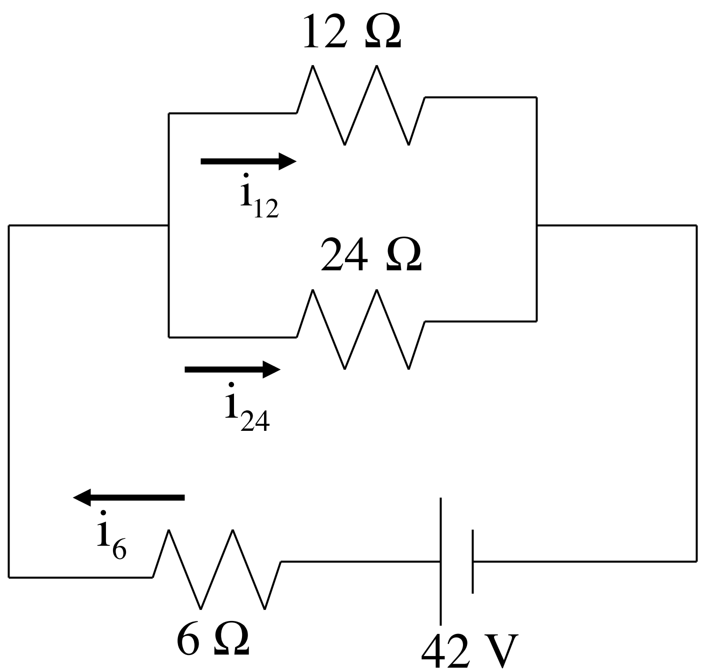
--And, if you add a resistor to the combination, R_{eq} will *decrease* and the current through the combination (for a given voltage) will *increase*.

Example 3: What's the equivalent resistance of three one-ohm resistors in parallel?

$$1/R_{eq} = 1/(1\Omega) + 1/(1\Omega) + 1/(1\Omega)$$

$$\Rightarrow 1/R_{eq} = 3 \Rightarrow R_{eq} = .333\ \Omega$$

Example 6: The current from the battery is 3 amps. How much current goes through the upper branch of the parallel combination?



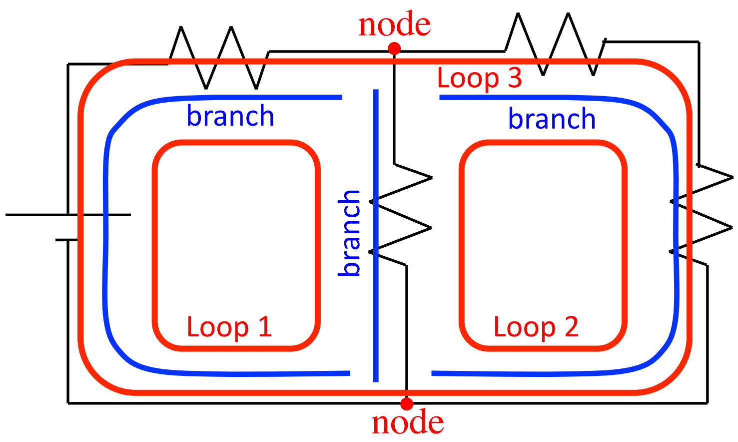
This is another use-your-head question.

If the upper branch has half the resistance of the lower branch, it should draw twice the current.

With 3 amps coming in, that means 2 amps should pass through the upper branch.

Note: AP questions often have easy, non-mathematical, use-your-head solutions like this. That is why I'm showing you screwball problems like this. We will get into a more formal approach for analyzing circuit problems shortly.

Some Definitions



A branch: A section of a circuit in which the current is the same everywhere.

--elements in series are a part of a single branch (look at sketch).

--in the circuit to the right, there are three branches.

A node: A *junction* where current can split up or be added to.

--elements in parallel have nodes internal to the combination.

--in the circuit above, there are two nodes.

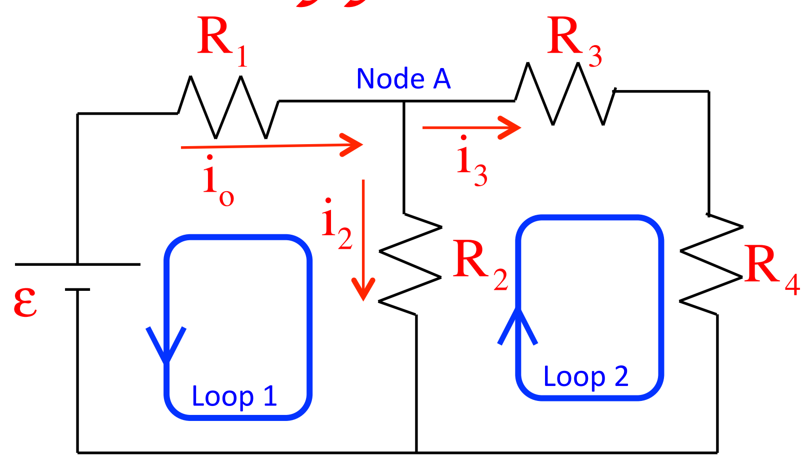
A loop: Any closed path inside a circuit.

--in a circuit, loops can be traverse in a clockwise or counterclockwise direction.

--in the circuit above, there are three loops.

Kirchoff's Laws—the Formal Approach

With the definitions under your belt, Kirchoff's Laws are simple (and you've been inadvertently using them in the seat-of-the-pants evaluations). They are:



Kirchoff's First Law: The sum of the currents into a node equals the sum of the currents out of a node. Mathematically, this is written as: $\sum i_{\text{into node}} = \sum i_{\text{out of node}}$

Example from the circuit's Node A: $i_o = i_2 + i_3$

Kirchoff's Second Law: The sum of the *voltage changes* around a closed path (a loop) equals ZERO. Mathematically, this is written as: $\sum \Delta V = 0$

Examples: starting at Node A:

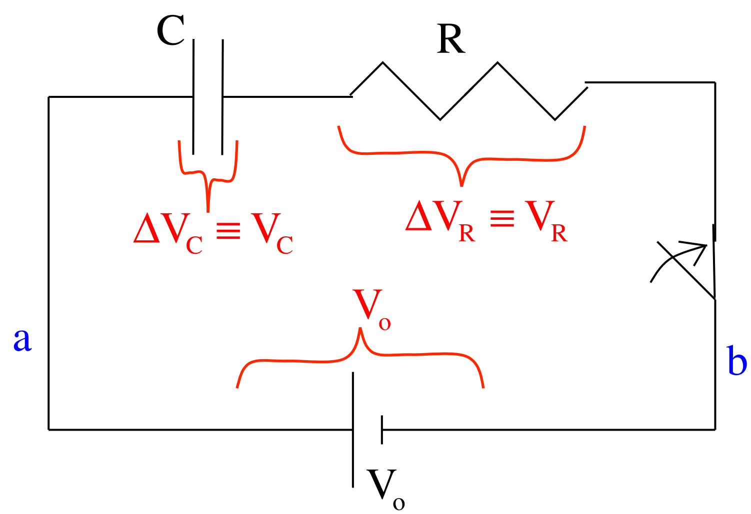
Loop 1 traversing counterclockwise: $R_1 i_o - \varepsilon + R_2 i_2 = 0$

Loop 2 traversing clockwise: $-R_3 i_3 - R_4 i_3 + R_2 i_2 = 0$

Note: Current moves from hi to lo voltage, so traversing against the current through a resistor produces a ΔV that is positive; traversing *with* current makes it negative.

Capacitors—Charging Characteristics

Example 10: Consider a resistor, an uncharged capacitor, a switch and a power supply all hooked in series. Note also that when the switch is thrown, the voltage across "a" and "b" is equal to both the *battery voltage* and the *sum of voltages across the resistor and capacitor*. That is:

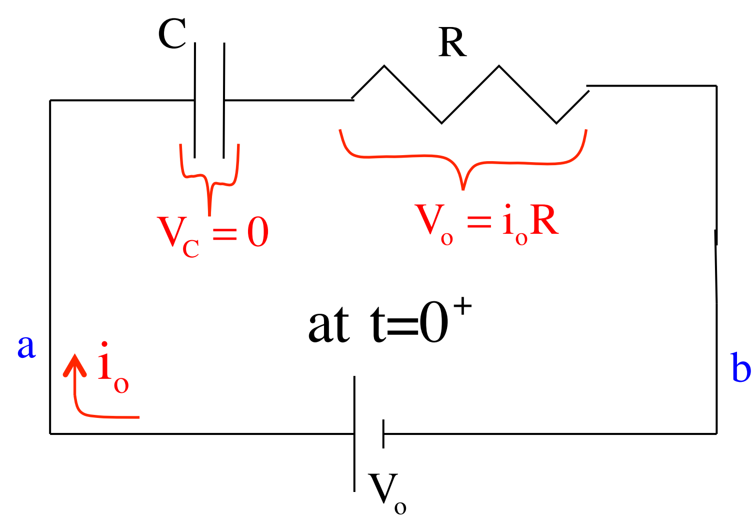


$$V_o = V_C + V_R$$

a.) At t = 0, the switch is closed. What initially happens in the circuit?

As the cap initially has no charge on its plates, it will provide no resistance to charge flow. That means *no voltage drop across the capacitor* with all the voltage drop happen across the resistor . . . which means:

$$V_o = \cancel{V_C}^{\,0} + V_R$$
$$= i_o R$$
$$\Rightarrow \quad i_o = \frac{V_o}{R}$$



Solving:

$$\frac{dq}{dt}+\left(\frac{1}{RC}\right)q=\frac{V_o}{R}$$

because $|a-b|=(b-a)$ if $b>a$.

$$\Rightarrow \quad \frac{dq}{dt}=\left(\frac{1}{RC}\right)(V_oC-q)=\left(\frac{1}{RC}\right)(Q_{max}-q)$$

$$\Rightarrow \quad \frac{dq}{(q-Q_{max})}=-\frac{dt}{RC}$$

$$\Rightarrow \quad \int_0^{q(t)}\frac{dq}{(q-Q_{max})}=-\int_{t=0}^{t}\frac{dt}{RC} \quad \Rightarrow \quad \ln|q-Q_{max}|\Big|_{q=0}^{q(t)}=-\frac{t}{RC}$$

$$\Rightarrow \quad \ln|q(t)-Q_{max}|-\ln|-Q_{max}|=-\frac{t}{RC} \quad \Rightarrow \quad \ln(Q_{max}-q(t))-\ln(Q_{max})=-\frac{t}{RC}$$

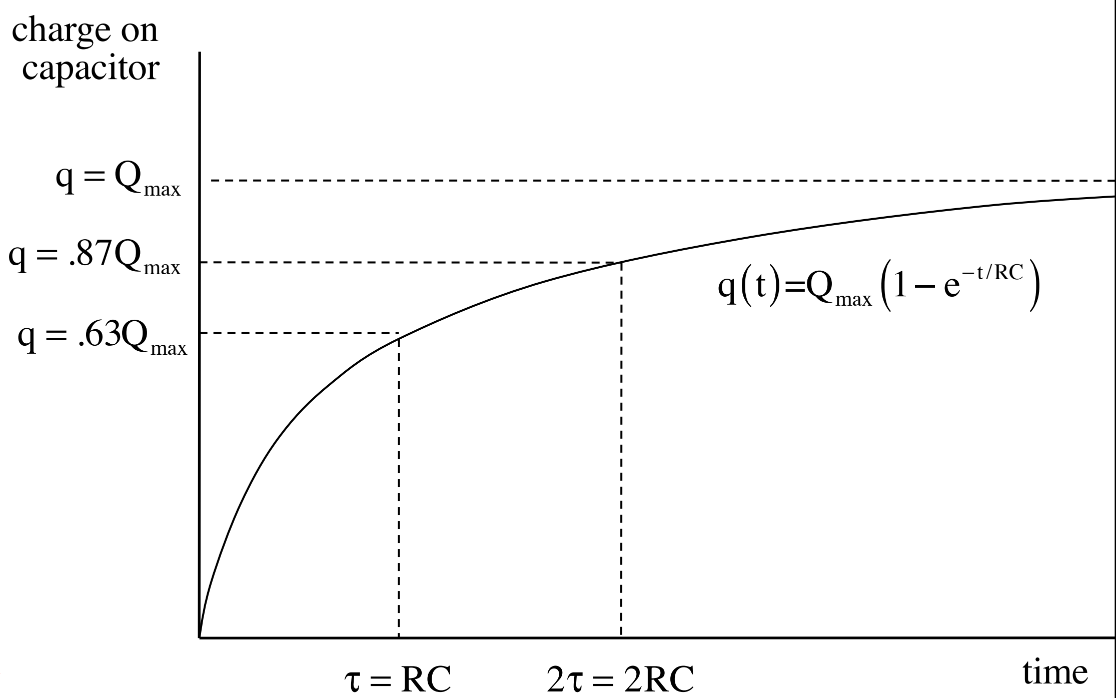
$$\Rightarrow \quad \ln\left[\frac{(Q_{max}-q(t))}{(Q_{max})}\right]=-\frac{t}{RC} \quad \Rightarrow \quad e^{\ln\frac{(Q_{max}-q(t))}{(Q_{max})}}=e^{-\frac{t}{RC}}$$

$$\Rightarrow \quad \frac{(Q_{max}-q(t))}{(Q_{max})}=e^{-\frac{t}{RC}} \quad \Rightarrow \quad Q_{max}-q(t)=Q_{max}e^{-\frac{t}{RC}} \quad \Rightarrow \quad q(t)=Q_{max}\left(1-e^{-\frac{t}{RC}}\right)$$

It would be nice to get a feel for how fast a capacitor/resistor combination will charge or discharge.

To that end, how much charge would the cap have accumulated after a time equal to *RC*?

$$q(t=RC) = Q_{max}\left(1 - e^{-\frac{RC}{RC}}\right)$$
$$= Q_{max}\left(1 - e^{-1}\right)$$
$$= Q_{max}\left(1 - \frac{1}{e}\right)$$
$$= Q_{max}\left(1 - .37\right) = .63Q_{max}$$

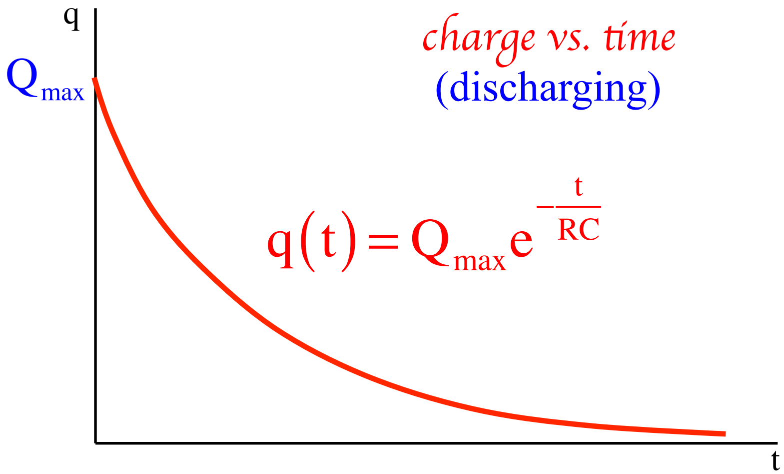


This time is defined as *one time constant* τ. It is the amount of time it takes the capacitor to charge to 63% of its maximum. *Two time constants* will charge it to 87% of its maximum (try the calculation if you don't believe me).

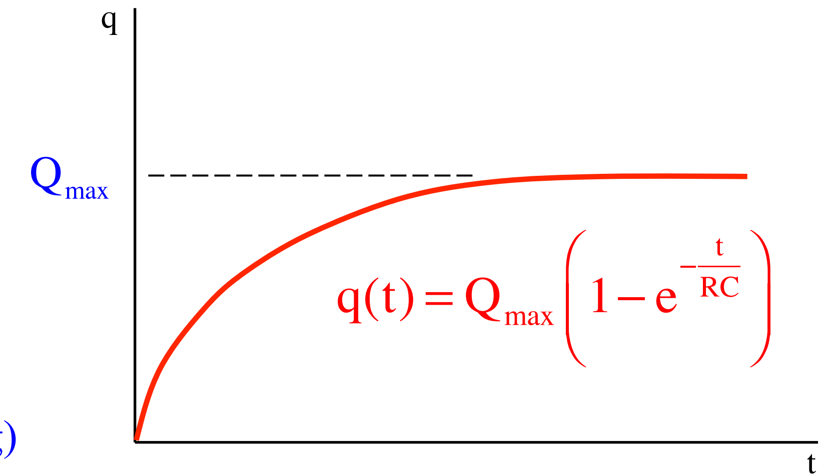
Summary of Graphs

Graphs of capacitor charging and discharging characteristics.

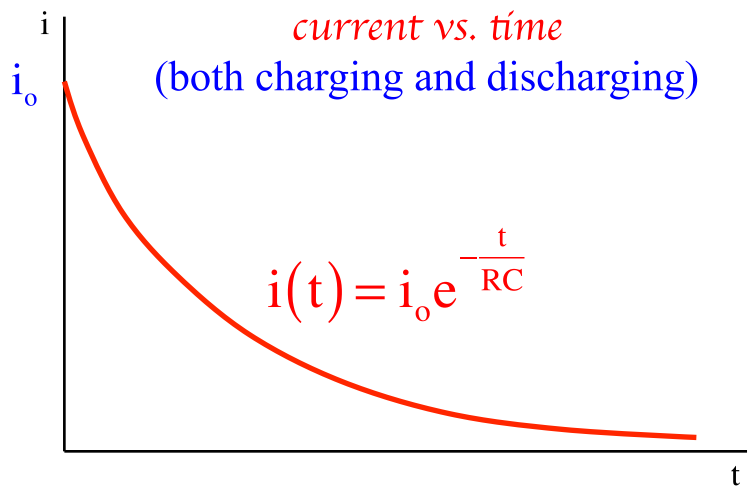
charge vs. time
(discharging)



charge vs. time
(charging)

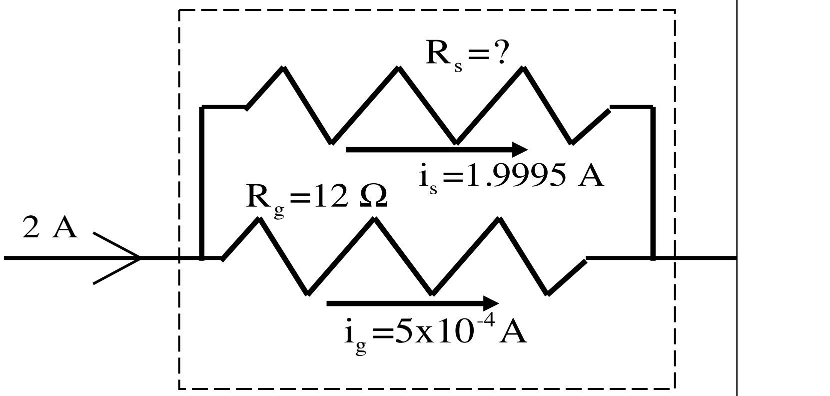


current vs. time
(both charging and discharging)

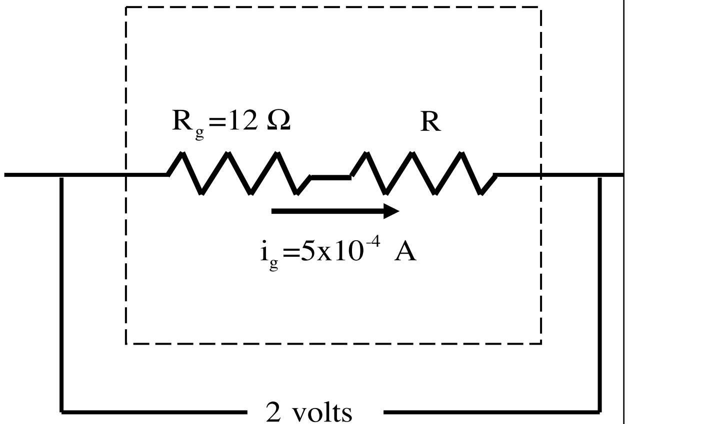


Observations:

--The galvanometer-engineered ammeter consists of a 12 Ω galvanometer in parallel with (in this case) a $3\text{x}10^{-3}\,\Omega$ resistor (that is, essentially a wire). As the equivalent resistance of a parallel combination is *smaller than the smallest resistor* in the combination, that means that the equivalent resistance of the ammeter is REALLY SMALL—exactly as expect.



--The galvanometer-engineered voltmeter consists of galvanometer and, in this case, an additional 40,000 Ω resistor in series. As the equivalent resistance of a series combination is *larger than the largest resistor* in the combination, that means that the equivalent resistance of the voltmeter is REALLY Big—again, exactly as expect.



General Information

Electric Fields

--*electric fields* (abbreviated as *E-flds*), with units of *newtons per coulomb* or *volt per meter*, are modified force fields (release a charge in an E-fld and it will accelerate);

--*electric fields* are generated with the presence of charge;

--*an electric field's direction* is defined as the *direction a positive charge will accelerate if released in the field*;

--*electric field lines:*

--*go from* positive to negative charge;

--*identify* the E-fld's direction in a region;

--*are closer together* where E-flds are more intense;

Magnetic Fields

--*magnetic fields* (abbreviated as *B-flds*), with units of *teslas* in the *MKS system*, are NOT modified force fields (release a charge in a B-fld and it will just sit there);

--*magnetic forces do exist* when a charge moves through a B-fld—they are centripetal and are governed by the relationship: $\vec{F} = q\vec{v}x\vec{B}$

--*B-fields* are generated by charge in motion;

--*a B-field's direction* is defined as the *direction a compass points when placed in the field*;

--*magnetic field lines:*

--*go from* north to south pole, or circle around current carrying wire;

--*identify* the B-fld's direction in a region;

--*are closer together* where B-flds are more intense;

Magnetic Fields

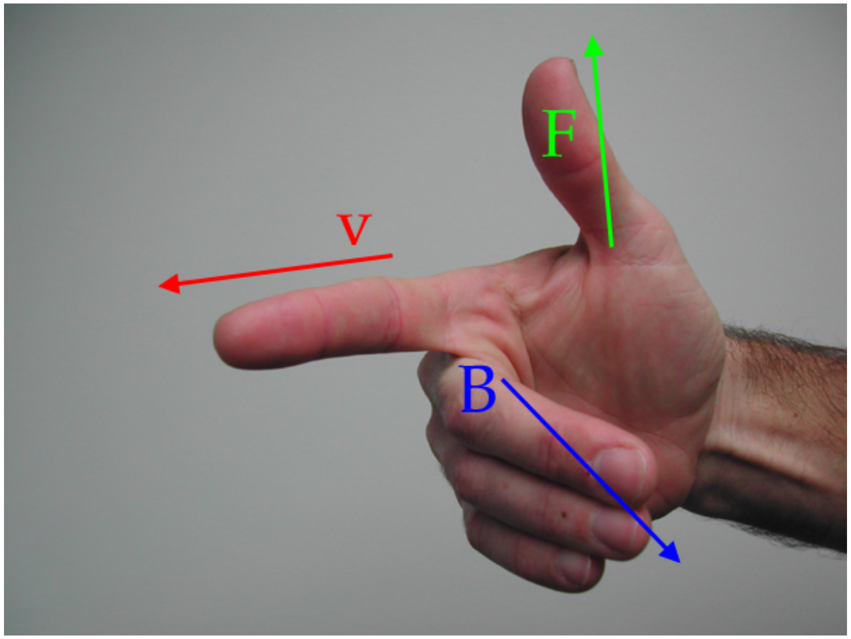
Magnetic Force

When charge moves through a magnetic field, it may or may not feel a force, depending upon its motion. If present, that force will be:

$$\vec{F} = q\vec{v}x\vec{B}$$

The magnitude is $|\vec{F}| = q|\vec{v}||\vec{B}| \sin\theta$, where q is the size of the charge, $|\vec{v}|$ is the magnitude of the velocity vector, $|\vec{B}|$ is the magnitude of the magnetic field and θ is the angle between the line of the two vectors.

The direction is determined using the *right-hand rule*.

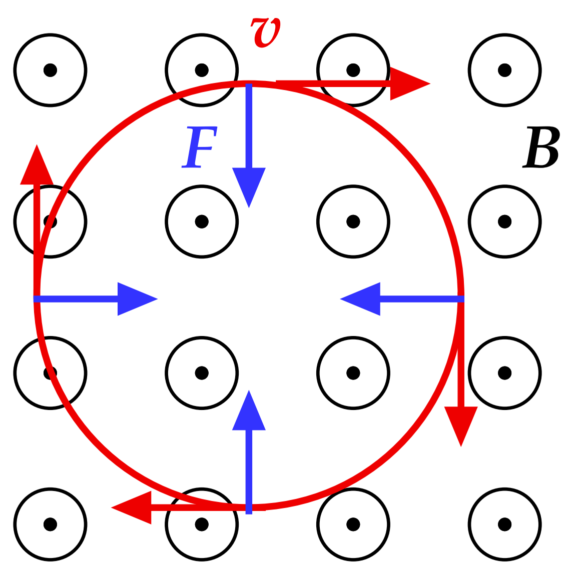


sketch courtesy of
Mr. White

Example 3: A charge q of mass *m* is moving with constant velocity *v* at right angles to a magnetic field *B*. (idea courtesy of Mr. White)

a.) What kind of motion will it execute?

Because magnetic forces are centripetal, the mass will follow a circular path.



b.) What is the radius of the motion's path?

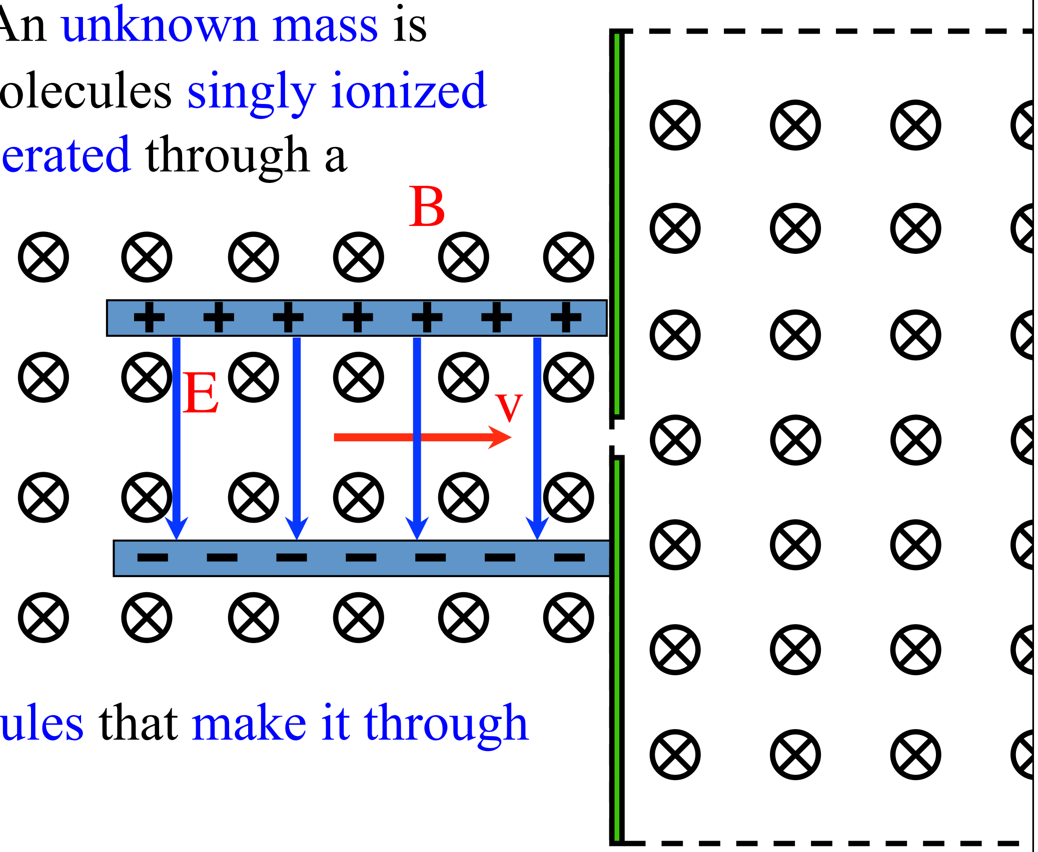
$$\sum F_{cent}:$$

$$q\vec{v}x\vec{B} = m\vec{a}_{cent}$$

$$\Rightarrow \quad qvB\sin 90^{o} = m\frac{v^2}{R}$$

$$\Rightarrow \quad R = \frac{mv}{qB}$$

Example 5 (mass spectrometer): An unknown mass is volatalizes (made into a gas), had its molecules singly ionized (had one electron stripped away), accelerated through a potential difference to give them velocity, and sent through a velocity trap made up of a 95,000 V/m *E-fld* and a .93 teslas *B-fld*. The molecules that make it through the trap move into a region in which there is only the *B-fld*.



a.) What is the *velocity* of the molecules that make it through the trap?

$$qE = qvB$$

$$\Rightarrow \quad v = \frac{E}{B} = \frac{9.5\text{x}10^4 \ \text{V/m}}{.93 \text{ T}}$$

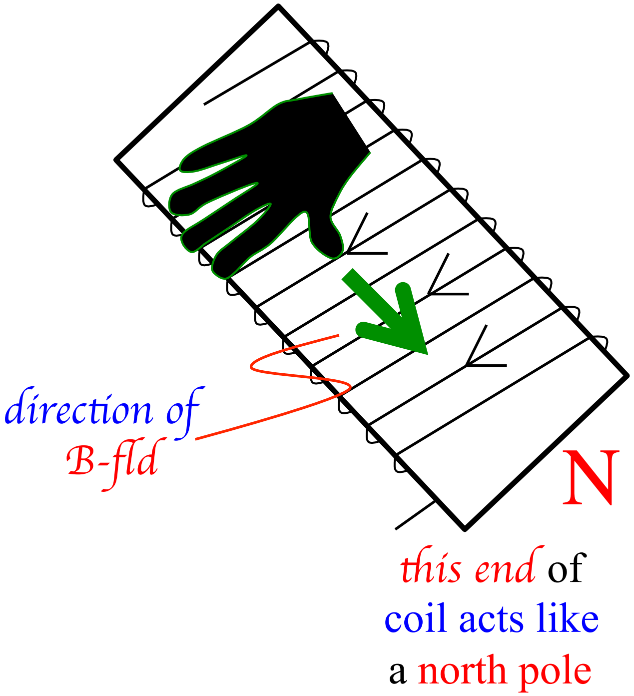
$$= 1.02\text{x}10^5 \text{ m/s}$$

Other Devices

A little more sophisticated version of a motor required one bit of information that would normally not be covered until next chapter.

It is charge in motion that generates magnetic fields. With current carrying coils, the generated magnetic fields are *down the axis* and *through the face of the coil*.

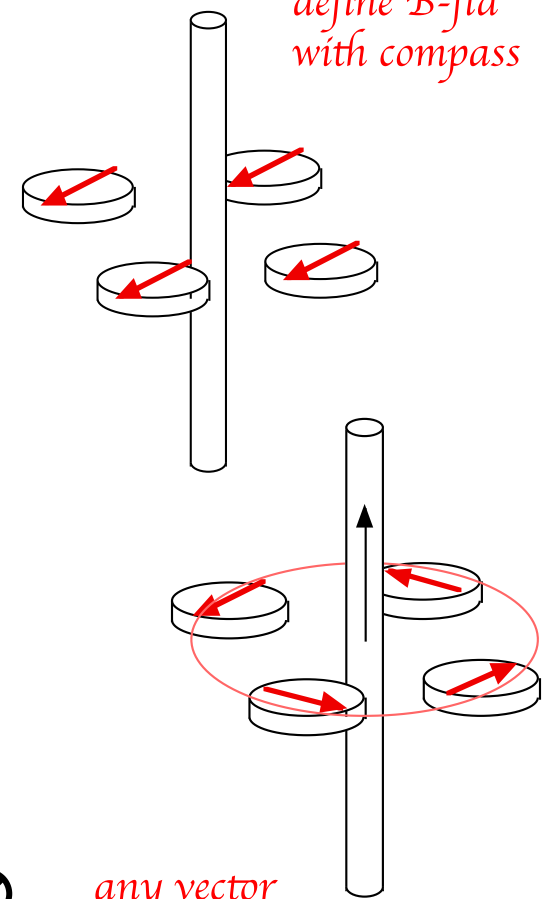
A handy trick to determine the direction of a current carrying wire's *B-fld* is to lay your right hand on the coil with your fingers following the direction of current in the coil. The direction in which your extended right-thumb points identifies the direction of the coil's B-fld.



Note that with the *B-fld* extending along the axis as it does, the coil's ends look like north and south poles.

With that, consider the following:

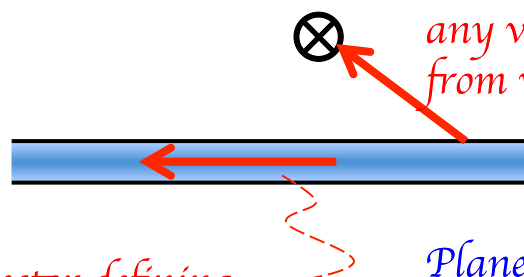
Oersted (1820) (courtesy of Mr. White)



If the wire is grasped with the right hand, with the thumb in the direction of current flow, the fingers curl around the wire in the direction of the magnetic field.

The magnitude of B is the same everywhere on a circular path perpendicular to the wire and centered on it. Experiments reveal that B is proportional to I, and inversely proportional to the distance from the wire.

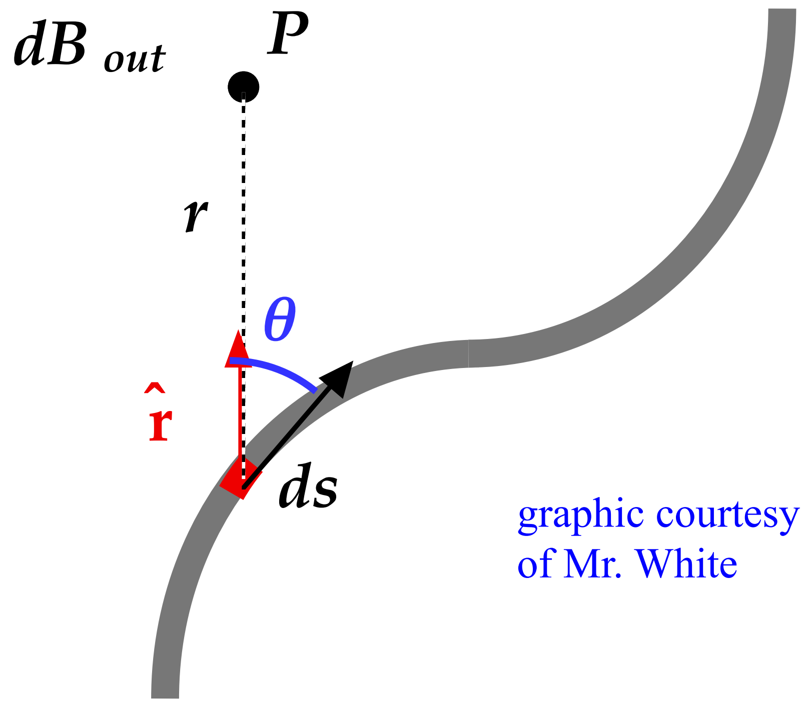
Obscure observation from Fletch: Notice that if the *current-carrying wire* is straight and you draw a vector from any point on the wire to a point of interest, the direction of the magnetic field at that point will be perpendicular to the plane defined by that vector and the direction of the current (treated like a vector).



Biot-Savart does a similar thing for *magnetic fields,* with the exception that it incorporates the *direction of the B-field* into the calculation.

Specifically, it observes that the differential magnetic field *dB* at Point P due to the current in the differentially small section *ds* of wire is:

$$d\vec{B} = \left(\frac{\mu_o}{4\pi}\right) I \frac{d\vec{s} \times \hat{r}}{r^2}$$



graphic courtesy of Mr. White

where:

I is the current in the wire

μ_o = permeability of free space
$= 4\pi x 10^{-7}\ T \bullet m/A$

ds is a section of current-carrying wire

$d\vec{s}$ is a vector in the direction of the current at ds

$\vec{r}$ is a vector from ds to the point of interest

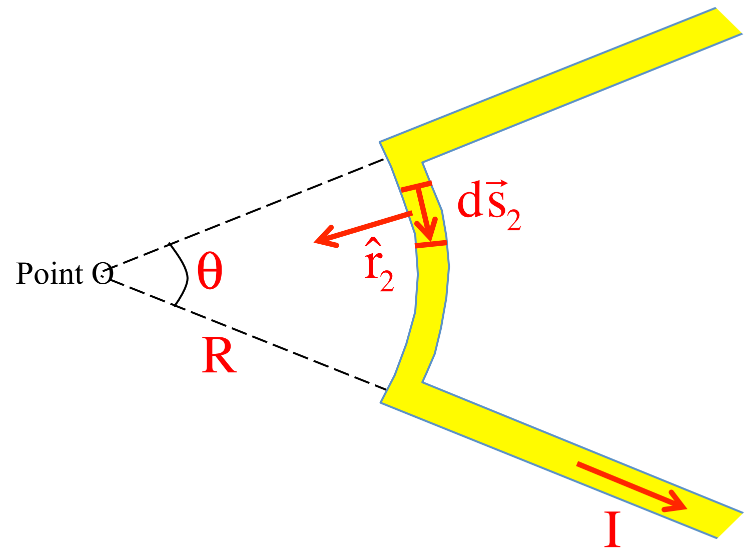
$\hat{r}$ is a unit vector in the direction of $\vec{r}$

θ is the angle between $\hat{r}$ and $d\vec{s}$

Notice the cross product gives a direction that is perpendicular to the plane defined by $\vec{r}$ and i at *ds* as advertised earlier.

With no B-flds being generated at Point O due to the sections of wire that have current moving directly toward or away from the point, we turn to the only other section in the system:

Again, defining the differential length *ds* and the unit vector $\hat{r}_1$ for the curved sections, and noticing how ds is related to $d\theta$ (see insert), the cross product becomes:



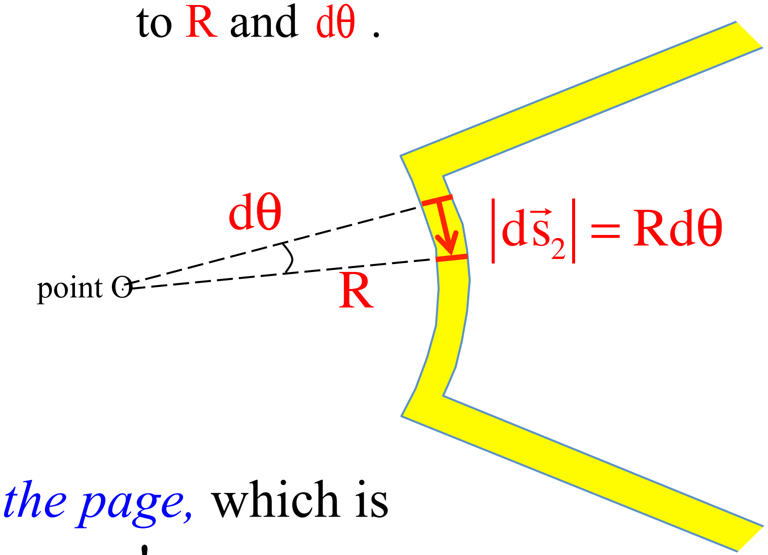
$$\left|d\vec{B}_1\right| = \left(\frac{\mu_o}{4\pi}\right) I \frac{\left|d\vec{s}_2 x \hat{r}_2\right|}{(R)^2} = \left(\frac{\mu_o}{4\pi}\right) I \frac{ds_2 \sin 90^o}{(R)^2}$$

$$\Rightarrow \quad B = \int dB = \left(\frac{\mu_o}{4\pi R^2}\right) I \int ds_2$$

$$= \left(\frac{\mu_o}{4\pi R^2}\right) I \int_0^{\theta} R\, d\theta$$

$$= \left(\frac{\mu_o I}{4\pi R}\right)\theta$$

How ds is related to R and $d\theta$.



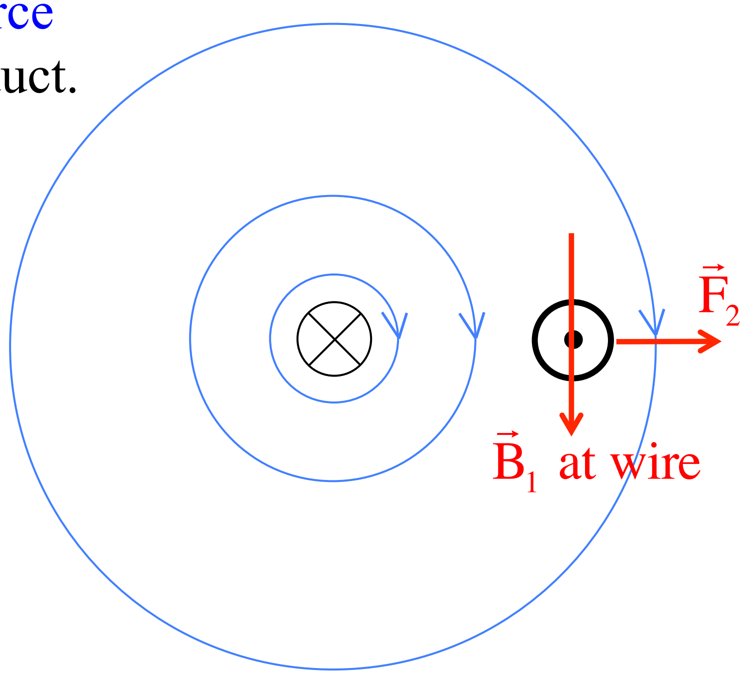
Also, crossing $d\vec{s}_2$ into $\hat{r}_2$ yields a direction *INTO the page,* which is exactly what the *right-thumb* rule would have given you!

Because the force relationship between a current-carrying wire and the *B-fld* the wire is bathed in is known, we can write:

$$\left|\vec{F}_2\right| = i_2\left|\vec{L}x\vec{B}_1\right|$$
$$= i_2 L\left(\frac{\mu_o i_1}{2\pi a}\right)$$

Now for the fun—finding the direction of the force on the right-hand wire: Start with the cross product.

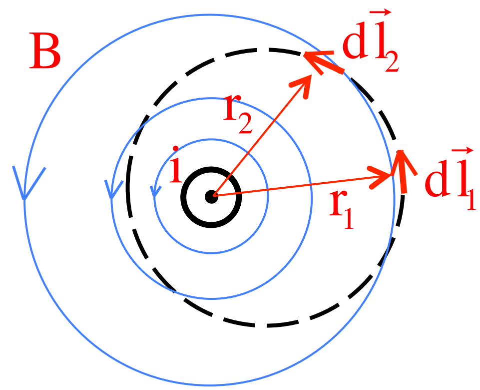
$$\vec{F}_2 = i_2\vec{L}x\vec{B}_1$$



$\vec{L}$ is out of the page (in the direction of the right-hand wire's current), and we've already determined the direction of the *B-fld* due to the left-hand wire in that region (it's downward at the right-hand wire).

Executing $\vec{L}x\vec{B}_1$ yields a vector direction to the *right,* AWAY from the *left wire.*

Example 6: A current carrying wire has current directed out of the page as shown. For the dotted path shown, is the net circulation equal to $\mu_o i$?



YES, Ampere's Law always works (just like Gauss's Law always works, even when a geometry makes its integral impossible to solve).

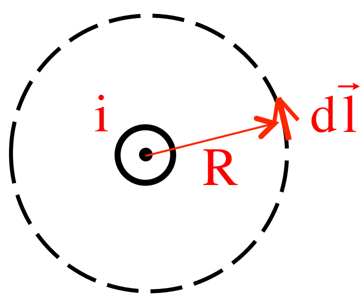
The real question is whether using Ampere's Law is a reasonable thing to try to do in this case . . . and the answer to that question is NO!

Why? Look at the symmetry.

The current through the face is easy—it's just *i*, but the angle between $d\vec{l}_1$ and the *B-fld* evaluated at $d\vec{l}_1$ is different than the angle between $d\vec{l}_2$ and the *B-fld* evaluated at $d\vec{l}_2$. That's going to make the integral nasty.

Consider the problem exploiting symmetry:

$|\vec{B}|$ is the same at every point on the path, so:



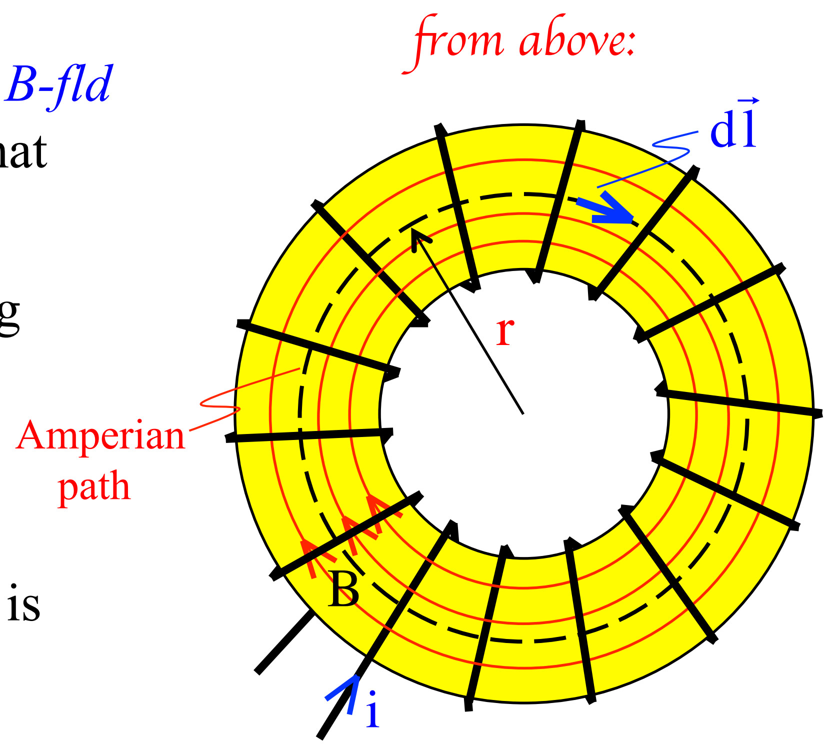
$$\oint \vec{B} \bullet d\vec{l} = \mu_o i_{thru}$$

$$\Rightarrow \quad B\oint dl \cancelto{1}{\cos 0^0} = \mu_o i$$

$$\Rightarrow \quad B(2\pi R) = \mu_o i \quad \Rightarrow \quad B = \frac{\mu_o i}{2\pi R}$$

BAM! The *B-fld* for a current-carrying wire.

Example 8: Derive an expression for the *B-fld* inside an N-turn toroid (a coil with N winds that curves back on itself)



--*Because the B-fld* for a toroid circles along the toroid's axis, the *Amperian path* that is applicable here is a *circle* of radius *r*.

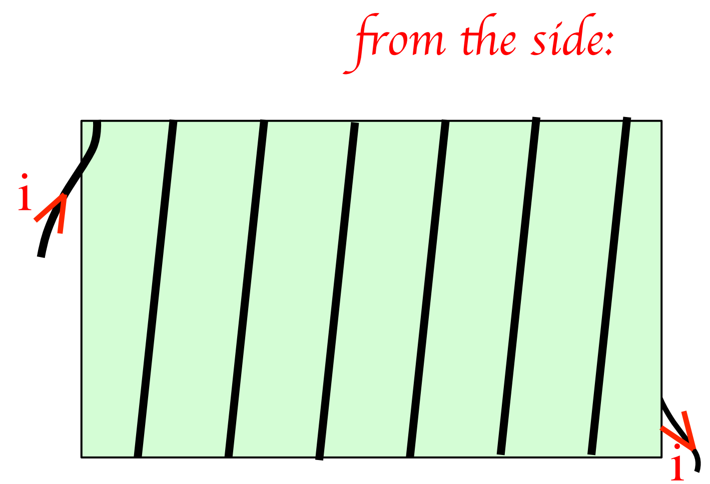
--*Noting that N wires* pass through the Amperian path, the current through the face is *Ni* and we can write:

$$\oint \vec{B} \bullet d\vec{l} = \mu_o i_{thru}$$

$$\Rightarrow \quad B\oint dl\,\cancel{\cos 0^o}^{\,1} = \mu_o (Ni)$$

$$\Rightarrow \quad B(2\pi r) = \mu_o (Ni)$$

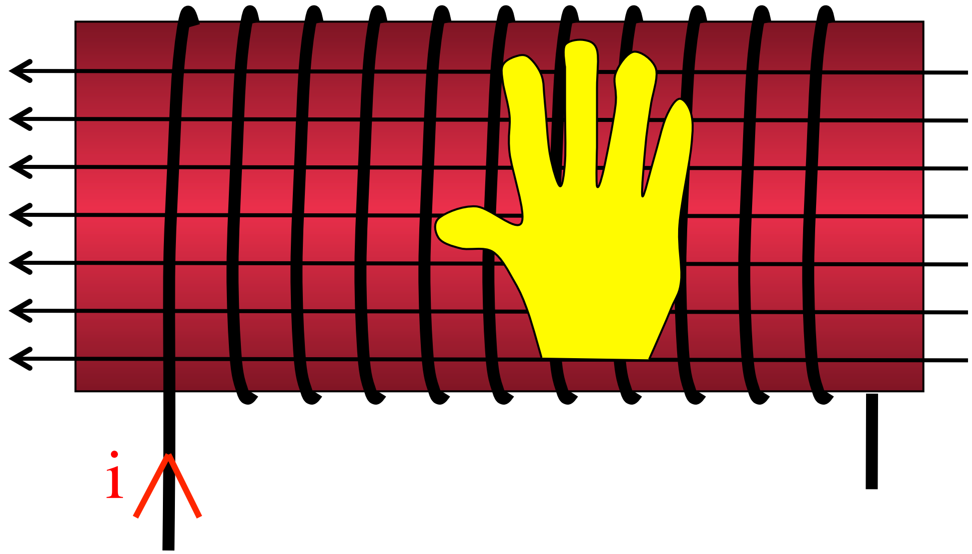
$$\Rightarrow \quad B = \frac{\mu_o Ni}{2\pi r}$$



--*Notice that B* varies with *r*.

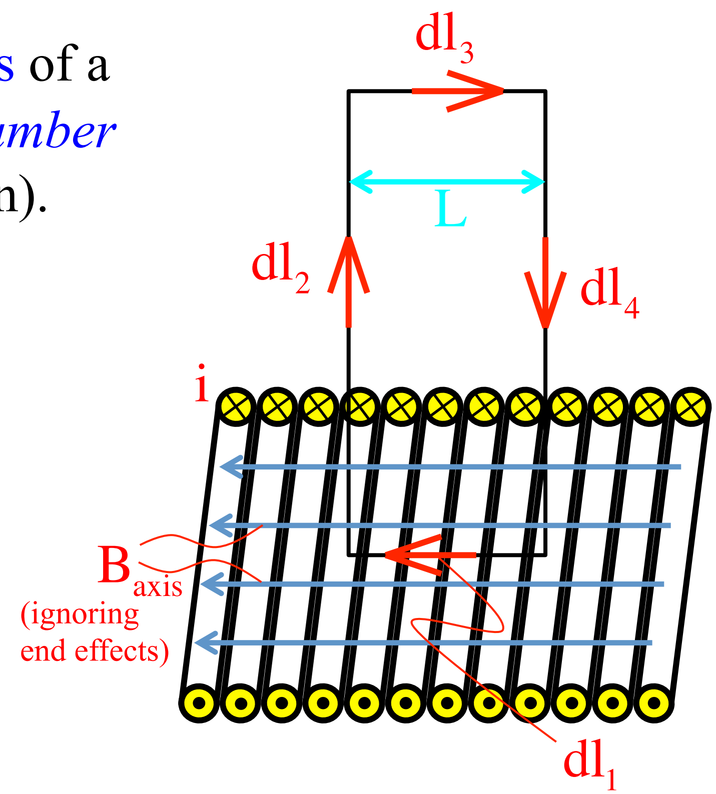
Trickery

There is still another right-hand rule that can be used to determine the direction of the magnetic field due to current through a coil. It's easy (and fun!).



Lay your right-hand on the coil with your fingers pointing in the direction of the current. The direction your thumb points is the direction of the *B-fld* down the axis of the coil.

Example 9: Determine the *B-fld* down the axis of a current-carrying coil (a solenoid), where *n* is the *number of turns per unit length* in the coil (see cross-section).



We need a rectangular *Amperian path*. Why?

--The paths perpendicular to the coil will experience zero *B-fld*;

--The path outside the coil is far enough out so the *B-fld* is essentially zero;

--The path inside the coil experiences a non-zero *B-fld*.

The current through the face is:

$$i_{thru} = (nL)i$$

where *nL* is the number of wires thru the face. So:

$$\oint \vec{B} \bullet d\vec{l} = \mu_o \, i_{thru}$$

$$\int_{S_1} \vec{B} \bullet d\vec{l}_1 + \int_{S_2} \vec{B} \bullet d\vec{l}_2 + \int_{S_3} \vec{B} \bullet d\vec{l}_3 + \int_{S_4} \vec{B} \bullet d\vec{l}_4 = \mu_o\left[(nL)i\right]$$

$$B_{axis}\int_L dl_1 \cancelto{1}{\cos 0^o} + \cancelto{0}{B_y}\int_h dl_2 + \cancelto{0}{B_{wayout}}\int_L dl_3 + \cancelto{0}{B_y}\int_h dl_4 = \mu_o\left[(nL)i\right]$$

$$\Rightarrow \quad B_{axis}\cancel{L} = \mu_o\left[(n\cancel{L})i\right]$$

$$\Rightarrow \quad B_{axis} = \mu_o n i$$

Deciding When to Use Ampere's Law versus Biot Savart

Use Ampere's Law when:

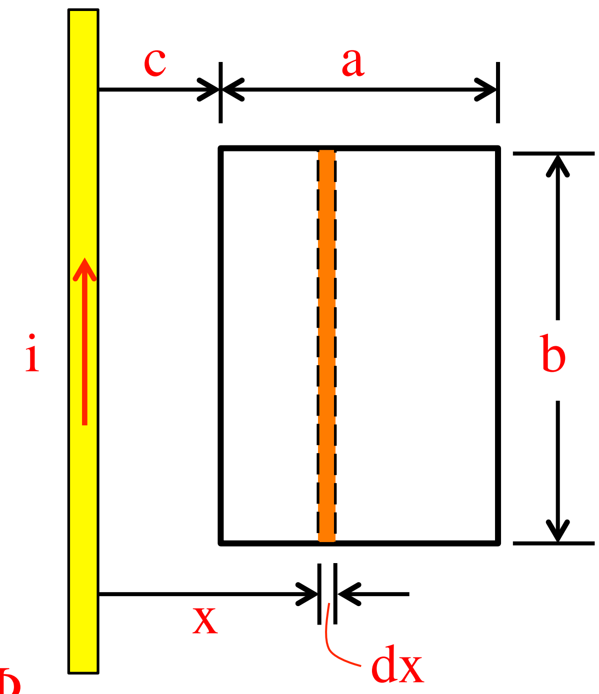
--*You can define* a path upon which the magnitude of *B* is constant over the entire path (this will normally be a circular path); or

--*You can define* a combination of paths some of which will have a magnitude of *B* that is constant over the section(s), some will have *B* equal to zero over the section(s) and/or some will have the evaluation of $\vec{B} \cdot d\vec{l}$ equal to zero over the section(s) . . . (these multiple paths are usually rectangular).

Use Biot Savart when:

--*You can't* use *Ampere's Law*. (In other words, Ampere's Law should be your first choice.)

Example 10: Derive an expression for the *magnetic flux* through the rectangular path shown due to the *B-fld* set up by the current-carrying wire (a very cool, classic problem).



The difficulty here is in the fact that the *B-fld* from the current carrying wire isn't constant over the face of the area, as $B_{wire} = \left(\frac{\mu_o}{2\pi}\right)\frac{i}{x}$ shows.

We have to determine the differential magnetic flux $d\Phi_B$ through a differentially small surface area $b(dx)$ where the *B-fld* is evaluated constant, then sum all those $d\Phi_B$'s over the entire face. Starting:

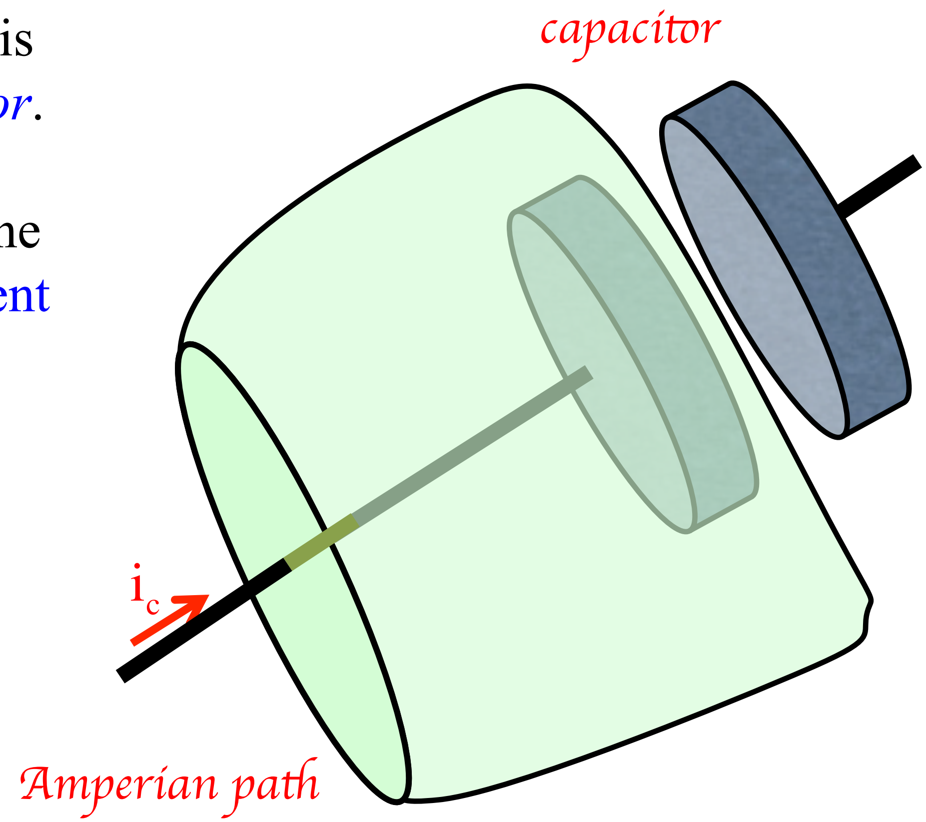
$$\begin{aligned} d\Phi_B &= \vec{B}\bullet d\vec{A} \\ &= \left(\frac{\mu_o i}{2\pi x}\right)(bdx)\cos 0^o \\ &= \left(\frac{\mu_o ib}{2\pi}\right)\frac{dx}{x} \end{aligned}$$

$\Rightarrow$

$$\begin{aligned} \Phi_B &= \int d\Phi_B \\ &= \left(\frac{\mu_o ib}{2\pi}\right)\int_{x=c}^{c+a}\frac{dx}{x} = \left(\frac{\mu_o ib}{2\pi}\right)\ln x\Big|_{x=c}^{c+a} \\ &= \left(\frac{\mu_o ib}{2\pi}\right)\left[\ln(c+a)-\ln c\right] = \left(\frac{\mu_o ib}{2\pi}\right)\left[\ln\left(\frac{c+a}{c}\right)\right] \end{aligned}$$

So $q_{encl} = \varepsilon_o \Phi_E$. But in this case, q_{encl} is the *charge on one plate of the capacitor*. If we calculate the rate at which that charge is changing (the rate at which the capacitor is charging), we get the current in the circuit. In other words:

$$i_c = \frac{dq_{cap}}{dt}$$
$$= \frac{d\left(\varepsilon_o \Phi_E\right)}{dt}$$
$$= \varepsilon_o \frac{d\Phi_E}{dt}$$



Caps don't have charge move through them, but the electrostatic repulsion between their plates creates the illusion that current is flowing through the cap. Faraday, apparently, deduced that that virtual current (my words, not his) was the displacement current needed for Ampere's Law to work. In any case, the complete form of Ampere's Law is:

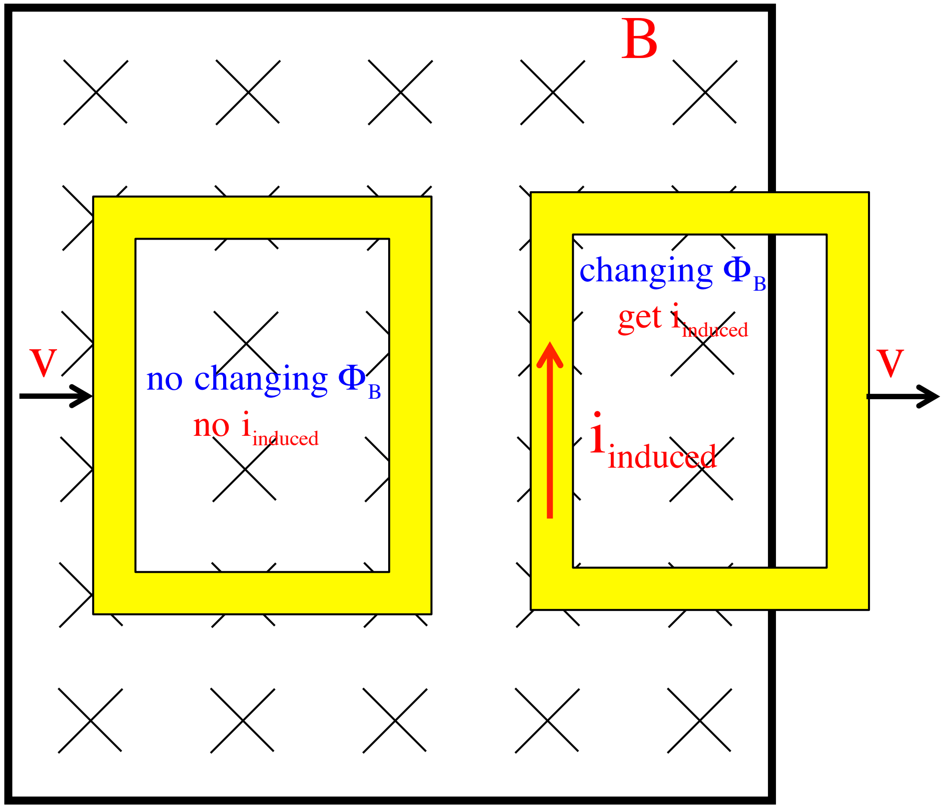
$$\oint \vec{B} \bullet d\vec{l} = \mu_o i_{thru} + \mu_o \left(\varepsilon_o \frac{d\Phi_E}{dt} \right)$$

where it's YOUR CHOICE which term *on the right* you evaluate, depending upon the circumstances.

Faraday's Law

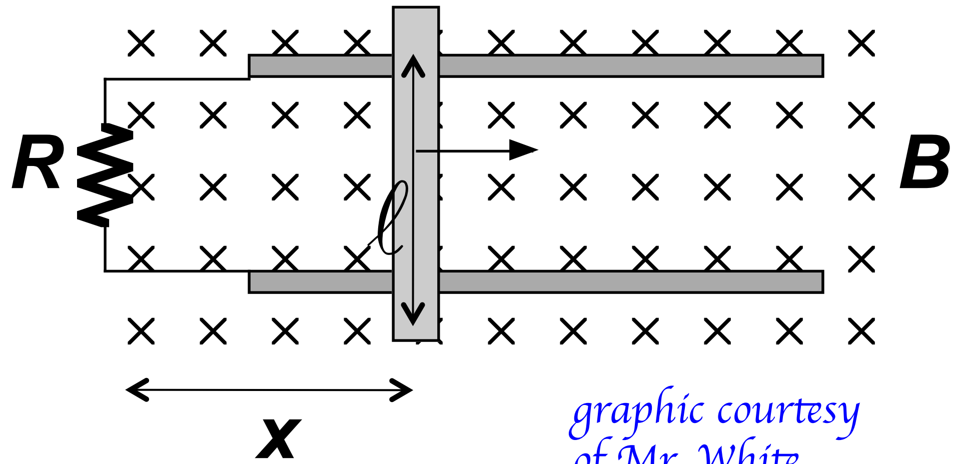
Faraday's Law

The creation of a conventional current flow as the coil leaves the constant *B-fld* has been explained using what you already know from the Classical Theory of Magnetism. Faraday viewed it differently. His approach will allow us to analyze difficult situations that are not so easily untangled with the thinking we've just presented.



Faraday, who was not interested in the dipole, noticed that you only get an induced current when there is a *changing magnetic flux* through the face of the coil. There could *be* a flux through the coil, but if it wasn't/isn't changing, no induced current.

Example 3: *A bar* on frictionless rails is made to move with velocity *v* through a *B-fld* as shown in the sketch (you are looking down on the system).



a.) Derive an expression for the induced EMF in the "coil."

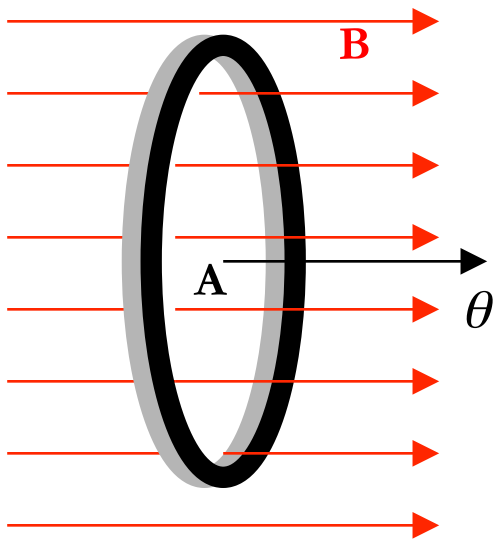
The technique here is to write out a general expression for the magnetic flux, then take its derivative. Doing so yields:

$$\Phi_B = \vec{B} \bullet \vec{A}$$
$$= BA\cancel{\cos 0^o}^{1}$$
$$= B(lx)$$

$$\Rightarrow \quad \varepsilon_{ind} = -\cancel{N}^{1}\frac{d\Phi_B}{dt}$$
$$= -\frac{d(Blx)}{dt}$$
$$= -Bl\frac{dx}{dt} \qquad (-Blv)$$

Lenz's Law

Although Faraday's Law allows us to determine the magnitude of the induced EMF set up by a changing magnetic flux through the face of a coil and, by extension, the magnitude of the induced current through the coil, it says nothing about the *direction* of the induced current set up by the EMF. Lenz's Law is designed to fill in that gap.



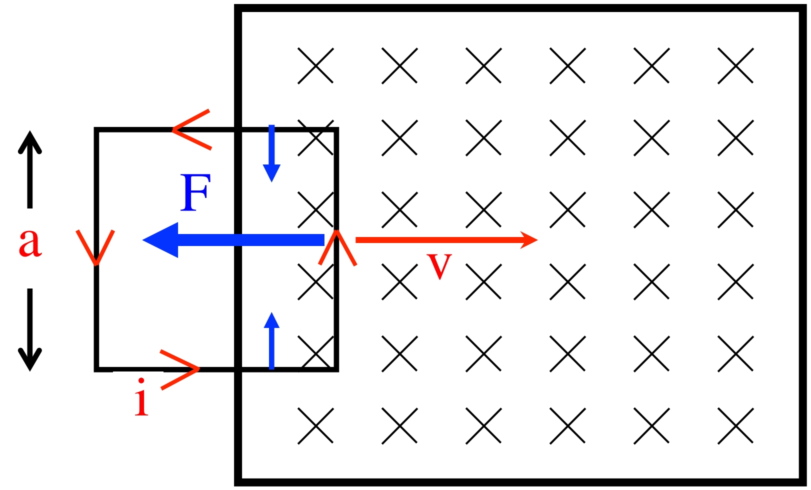
graphic courtesy of Mr. White

Lenz's Law maintains that an induced EMF through a coil (or loop) will produce an induced current that will create its own induced magnetic flux, and that that induced magnetic flux will *oppose the change* of magnetic flux through the loop that started the process off in the first place.

Confused? That's the statement of Lenz's Law in the raw. Its message can be more economically unpacked with three easy steps.

c.) What is the induced current in the coil?

$$i_{ind} = \frac{\varepsilon_{ind}}{R}$$
$$= \frac{Bav}{R}$$



d.) What is the direction of the current?

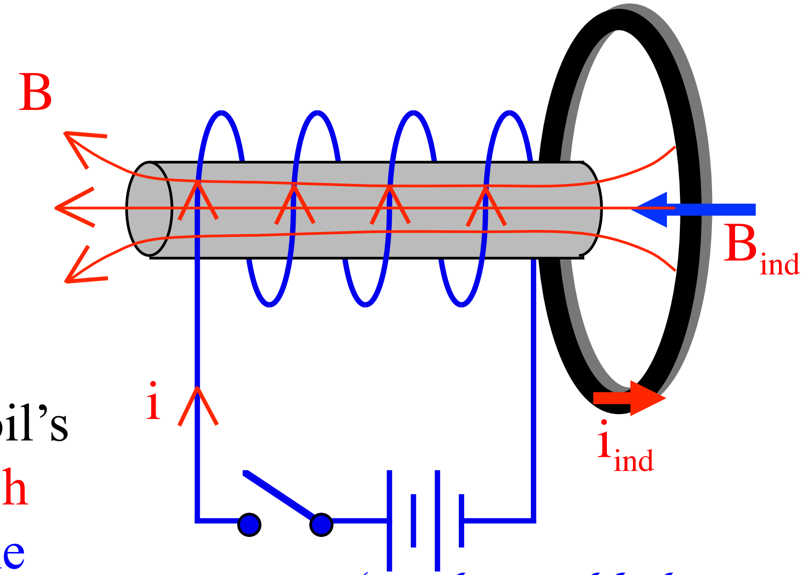
Lenz's Law:

--external *B-fld* into the page;
--magnetic flux increasing,
--so induced *B-fld OUT OF PAGE* (opposite external field). Current has to flow counterclockwise to achieve that.

e.) The induced current will interact with the external *B-fld* and feel a force. In what direction will be that net force?

The magnitude would be the magnitude of $\vec{F}_{wire} = i\vec{L}x\vec{B}$, which we could figure out, but all that was asked for was the direction, which is the direction of that cross product. The force on the two horizontal wires will cancel, but the force on the vertical wire in the *B-fld* will be to the left, as shown on the sketch.

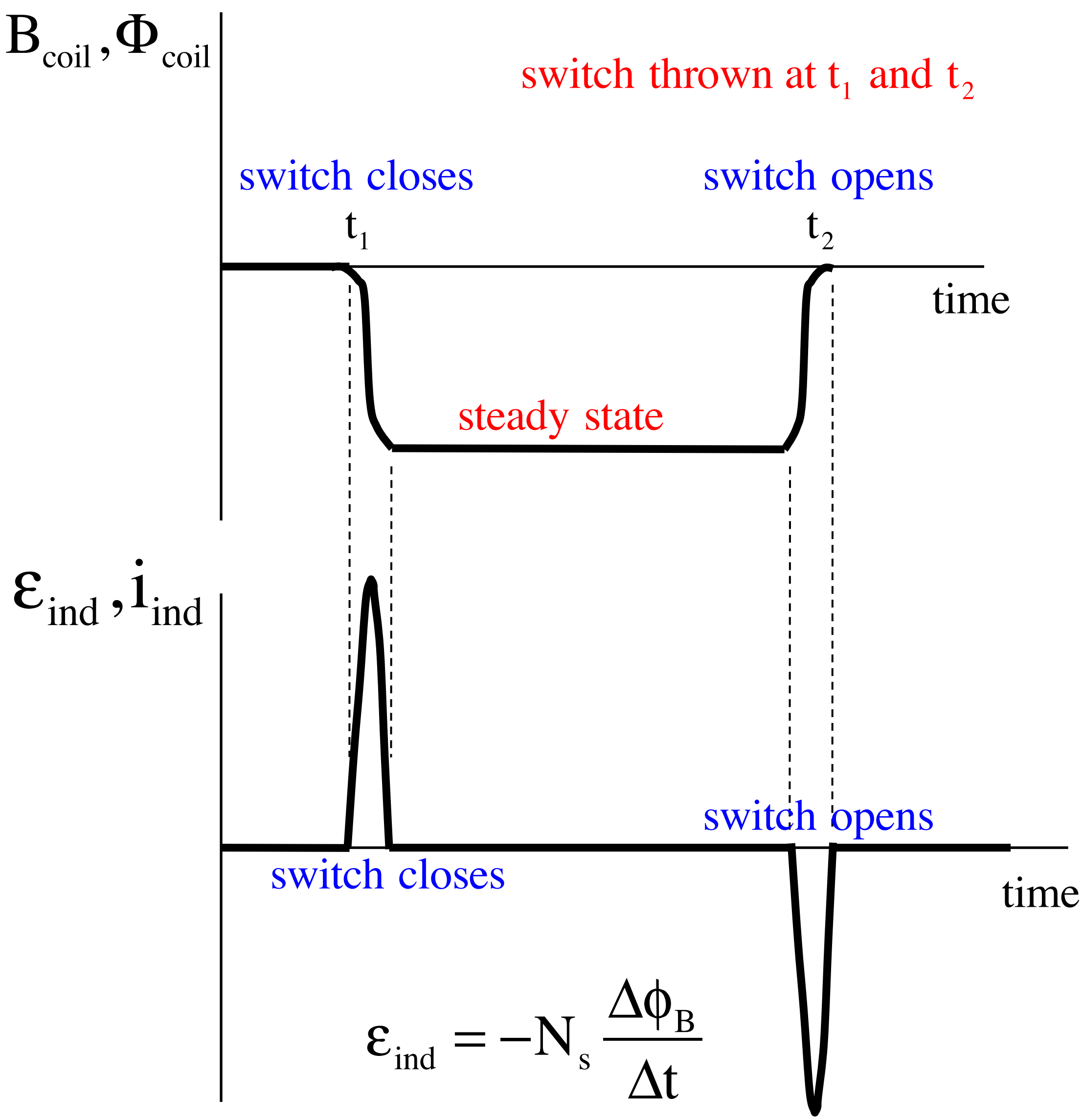
c.) Is there an induced current in the secondary coil when the switch is opened after being closed for a long time? If so, in what direction will the current be?



(graphic, modified, courtesy of Mr. White)

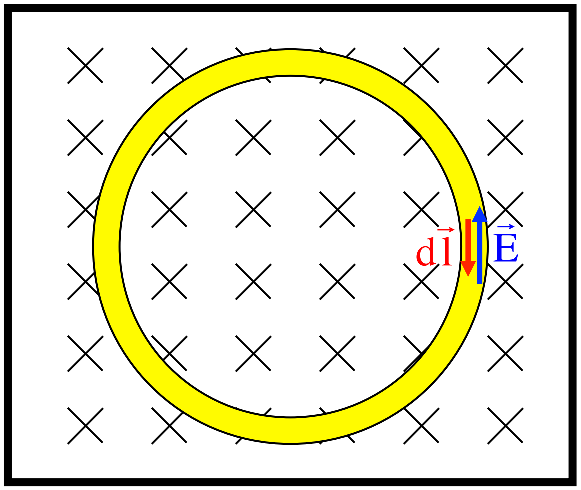
The direction of the coil's B-fld down the coil's axis won't changed, but now it will diminish to zero. That means the induced EMF in the secondary coil will produce an induced *B-fld* that is in the SAME DIRECTION AS the external field, or to the left. That will require a counterclockwise induced current.

d.) What will the graph of the current in the second coil look like as a function of time?



Example 12: For the situation shown

a.) Derive an expression for the magnitude of the *E-fld* should the *B-fld* increase.



Lenz's Law maintains the induced current, hence the induced *E-fld*, will be counterclockwise. With the area vector in the direction of the external magnetic field and $d\vec{l}$ appropriately defined (see previous slide for explanation, and see sketch for result), we can conclude:

The external B-fld is *into the page,* so the angle between that *B-fld* and the *area vector* will be 0°; the angle between $\vec{E}$ and $d\vec{l}$ is 180°, so we can write:

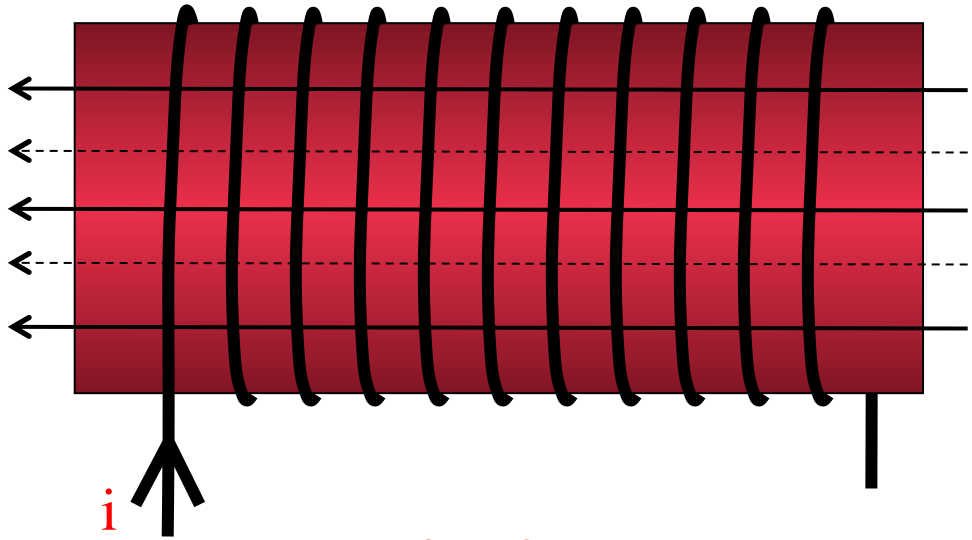
$$-\frac{d\Phi_B}{dt} = \oint \vec{E} \bullet d\vec{l}$$

$$\Rightarrow \quad -\frac{d\left[B\left(\pi R^2\right)\cos 0^\circ\right]}{dt} = E\oint dl\cos 180^\circ$$

$$\Rightarrow \quad -\cancel{\pi}R^2\left(\frac{dB}{dt}\right) = -E\left(2\cancel{\pi}R\right) \quad \Rightarrow \quad E = \frac{R}{2}\left(\frac{dB}{dt}\right)$$

Notice: If the external *B-fld* was decreasing, *dB/dt* would be negative making *E* negative. That would tell us the *E-fld* was opposite originally assumed!

Example 16: Derive an expression for the inductance of a solenoid of length *l*, radius *r* and total number of turns *N*.



$$
\begin{aligned}
\Phi_B &= B_{coil} A \cos 0^0 \\
&= B_{coil} A \\
&= (\mu_o n i)(\pi r^2) \\
&= \left(\mu_o \frac{N}{L} i\right)(\pi r^2)
\end{aligned}
$$

$$L\frac{di}{dt} = N\frac{d\Phi_B}{dt}$$

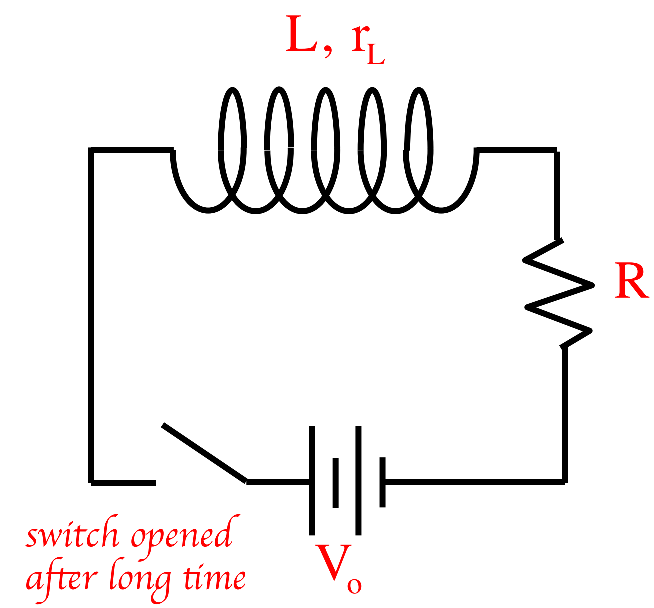
$$
\begin{aligned}
\Rightarrow \quad L &= N\frac{\Phi_B}{i} \\
&= N\frac{\left(\mu_o \frac{N}{L} i\right)(\pi r^2)}{i} \\
&= \frac{\mu_o N^2 \pi r^2}{L}
\end{aligned}
$$

If we let n be the *number of winds per unit length* (i.e., *N/L*), and noting the volume of the coil is $\pi r^2 L$, we can write:

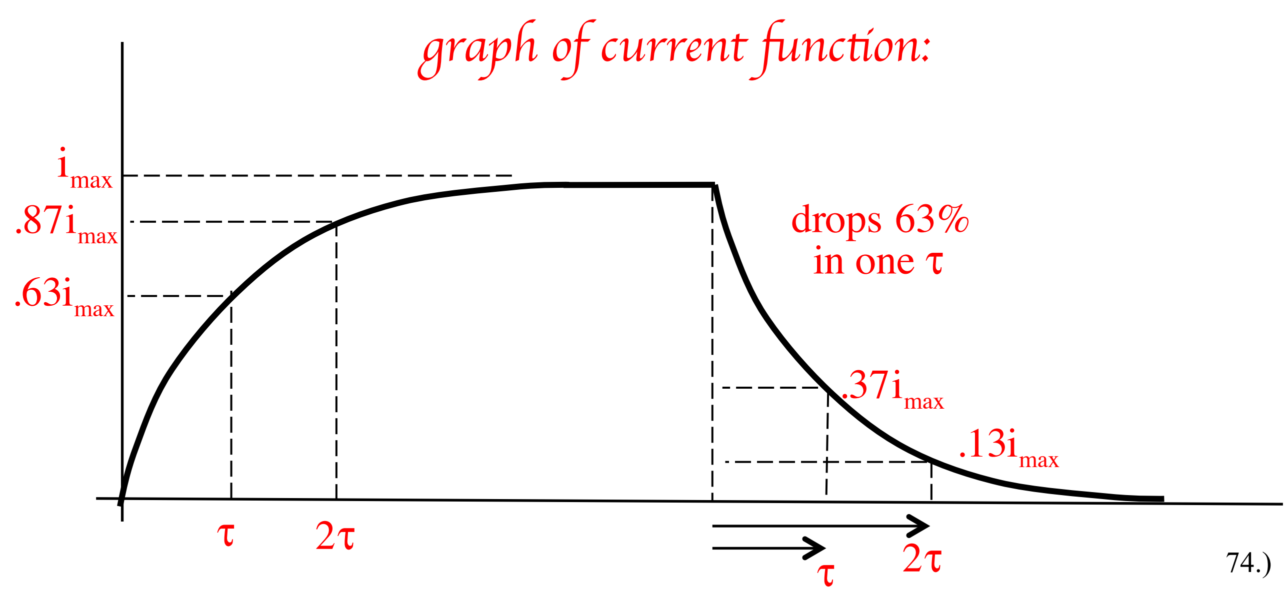
$$
\begin{aligned}
L &= \frac{\mu_o N^2 \pi r^2}{L} \\
&= \frac{\mu_o (n\cancel{L})^2 \pi r^2}{\cancel{L}} \\
&= \mu_o n^2 V
\end{aligned}
$$

f.) If current has been flowing for a long time, what happens when you open the switch?

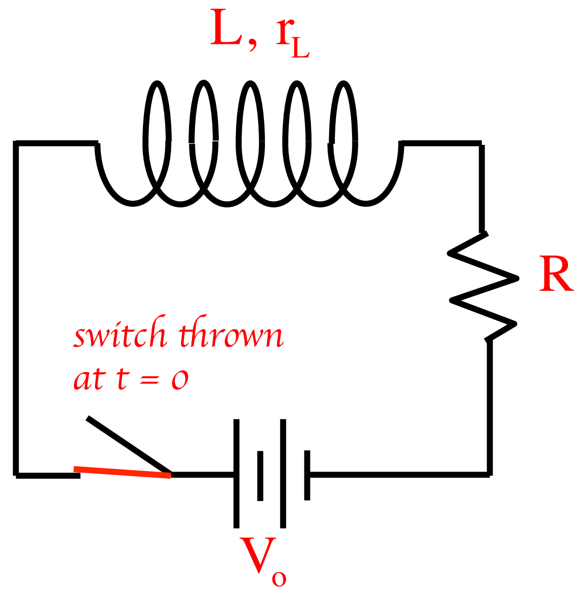
An attempted drop in battery-current will instigate an attempted drop in *B-fld* down the axis of the coil. That will induce an EMF that *fights the change*, which in this case means it will force current to flow *even longer than it normally would*. Due to the symmetry of the situation, it will take *one time constant* for the current to drop 63% of its maximum.



g.) The switch is closed, then after a long time it is opened. What will a graph of the *current vs time* look like for the system?



Knowing the power rating of an inductor, we can write:



$$P = \frac{dW}{dt} = \frac{-dU}{dt}$$

$$= -Li\,{}^{di}\!/_{dt}$$

Note: The *negative sign* in the second line is due to the fact that the power *stored* in an inductor (versus the power *dissipated* by an inductor) will (using Faraday's Law) be:

$$i\varepsilon = i\left(-L\frac{di}{dt}\right)$$

Continuing:

$$\frac{-dU}{dt} = -Li\,{}^{di}\!/_{dt}$$

$$\Rightarrow \quad dU = (Li)\,di$$

$$\Rightarrow \quad \int dU = L\int_{i=0}^{i} i\,di$$

$$\Rightarrow \quad U_L = \frac{1}{2}Li^2$$